Valuation of Derivative Contracts Using Payoff Event Approximation

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I Introduction

The number and variety of financial derivative contracts has expanded enormously since the opening of the option exchanges a quarter century ago. This has kept financial economists busy trying to derive formulae to price and hedge the increasingly complex contracts. Often no closed-form solution can be found, and pricing must rely on some form of approximation like finite differences or a numerical integration.

This paper proposes a semi-analytical approach — event approximation. Event approximation is related to numerical integration, but should provide faster and more accurate answers since the method combines analytical formulae so as to reduce the numerical work and eliminate errors.

Using the Cox-Ross technique an option can be priced by discounting the risk-neutral expectation of its payoff at the interest rate. If a formula cannot be determined, the risk-neutral expectation can be approximated by a numerical integration of the payoff with respect to the risk-neutral probability distribution or a by Monte Carlo estimation. The error in numerical integration arises from approximating the integrand over a small region by its value at a point in the region. Additional error may be due to truncating the integration thereby ignoring the tails of the distribution. The error in Monte Carlo approaches is similarly due to a histogram-like representation of the continuous distribution plus sampling error in values for the histogram.

Rather than approximating the integrand (the payoff times the probability), the event approximation method computes the expected value of the payoff for a set of events which approximate the actual payoff events. In terms of a numerical integration, this means that the approximating integrand exactly matches the true integrand over most of the domain.

By an appropriate selection of the payoff events, event approximation can be made to yield both upper and lower bounds for the value of the contract. This calibrates the accuracy of the approximation. In almost all cases, these bounds can be made as tight as is desired.

The payoff approximation method is particularly effective for pricing contracts based on two or more underlying assets. Such "rainbow" contracts often do not have analytical solutions because the distribution for the sum or difference of two lognormal variables is unknown. The payoff approximation method is described in the next section and illustrated with examples.

II Valuation by Event Approximation

Consider a call option written on a spread between two assets, S_1 and S_2. The payoff on this option is $\text{Max}[S_{1T} - S_{2T} - X, 0]$. Using the Cox-Ross technique, this option's value can be expressed as

$$e^{-r(T-t)}\hat{\mathrm{E}}[\mathrm{Max}(S_{1T} - S_{2T} - X,\ 0)] = e^{-r(T-t)} \iint_{S_{1T}-S_{2T}>X} (S_{1T} - S_{2T} - X)\mathrm{d}\hat{F}(S_{1T}, S_{2T}) \quad (1)$$

where $\hat{\mathrm{E}}$ is the expectation with respect to the risk-neutral probability distribution, $\hat{F}$. If the asset prices are lognormally distributed, the usual assumption, this value cannot be determined in closed form since the distribution of the difference between two lognormal random variables is an unsolved problem in statistics.

An approximate value for this option can be determined by using in place of the expectation the numerical integration

$$\hat{\mathrm{E}}[\mathrm{Max}(S_{1T} - S_{2T} - X,\ 0)] \approx \sum_{i=1}^{I}\sum_{j=1}^{i} (i\cdot\Delta S_1 - j\cdot\Delta S_2 - X)\mathrm{d}\hat{F}(i\cdot\Delta S_1, j\cdot\Delta S_2)\cdot\Delta S_1\cdot\Delta S_2 \quad (2)$$

where I is some sufficiently large number as suggested by Pearson [1995].

The method is characterized in Figure 1. The domain of positive payoff is divided into many small regions. For each region a value of the probability and the payoff are determined, and these values are summed to approximate the integration. More sophisticated numerical integration procedures, such as using an interpolated value of the integrand in each region, can be employed, but they all share this same basic approach.

The error in this approximation comes from three sources: (i) Implicitly it has been assumed that the density function and payoff are constant throughout each small region of size $\Delta S_1 \times \Delta S_2$. (ii) The payoffs for very large values of the asset prices, $S_{1T} > I\cdot\Delta S_1$ have been ignored.[1] (iii) The payoffs for some barely in-the-money outcomes, $i\cdot\Delta S_1 < S_{2T} - X < (i+1)\cdot\Delta S_1$, have been ignored. Since the payoff for this option is always nonnegative, the second and third types of errors will lead to underpricing. The first type of error has an indeterminate effect on the error in general.[2]

A Monte Carlo valuation is similar in many respects. The expectation is replaced by a sample mean

[1] Numerical integration can use a change of variable, such as $z = 1/S$, so that infinite values of S are represented by finite values of z to eliminate the tails; however, this just substitutes a different problem namely very much larger areas are now represented by a single point.

[2] In some cases, for example if the product of the payoff and the probability is monotone, we may be able to sign the first type of error.

$$\hat{\mathrm{E}}[\mathrm{Max}(S_{1T} - S_{2T} - X,\ 0)] \quad \approx \quad \frac{1}{N}\sum_{n=1}^{N} \mathrm{Max}(S_1 \cdot \epsilon_1^n - S_2 \cdot \epsilon_2^n - X,\ 0) \tag{3}$$

where ϵ_1^n and ϵ_2^n have the appropriate joint lognormal distribution. Monte Carlo techniques avoid the errors of type (ii) and (iii) above since any outcome is allowed; however, they replace the true risk-neutral probability distribution with a sampled one so they use an incorrect value of the integrand at each point just like numerical integration.

Event approximation relies upon the exact evaluation of integrals like (1) for events *similar* to the payoff event. In particular, since most derivative contracts have payoffs which are piece-wise linear in the underlying asset prices we want to compute

$$\begin{aligned}\hat{\mathrm{E}}_\Omega[a_1 S_{1T} + a_2 S_{2T} + b)] &= a_1 \iint_\Omega S_{1T}\,\mathrm{d}\hat{F}(S_{1T}, S_{2T}) + a_2 \iint_\Omega S_{2T}\,\mathrm{d}\hat{F}(S_{1T}, S_{2T}) + b \iint_\Omega \mathrm{d}\hat{F}(S_{1T}, S_{2T}) \\ &\approx a_1 \iint_{\Omega'} S_{1T}\,\mathrm{d}\hat{F}(S_{1T}, S_{2T}) + a_2 \iint_{\Omega'} S_{2T}\,\mathrm{d}\hat{F}(S_{1T}, S_{2T}) + b \iint_{\Omega'} \mathrm{d}\hat{F}(S_{1T}, S_{2T})\end{aligned} \tag{4}$$

where Ω' is chosen to be an event (or a union of sub-events) similar to Ω for which the integrals can be computed. The value of the contract can therefore be approximated as the present value of this expectation

$$\begin{aligned}&e^{-r(T-t)}\hat{\mathrm{E}}_\Omega[a_1 S_{1T} + a_2 S_{2T} + b)] \\ &\quad\approx a_1 \mathcal{S}_1(S_1, S_2, t; T; \Omega') + a_2 \mathcal{S}_2(S_1, S_2, t; T; \Omega') + b\,\mathcal{D}(S_1, S_2, t; T; \Omega') \ .\end{aligned} \tag{5}$$

Here $\mathcal{D}(S_1,\ S_2,\ t;\ T;\ \Omega')$ is the present value of \$1 received at time T if the event Ω' occurs. Such a contract is called a digital or binary option.[3] A digital option can be thought of as a kind of risky zero-coupon bond. The bond pays \$1 if event Ω' occurs and defaults completely otherwise. The related contract $\mathcal{S}_i(S_1,\ S_2,\ t;\ T;\ \Omega')$ is called a digital share. At maturity, T, it pays (or converts into) one share of asset i if the event Ω' has occurred. Note that in general the value of each digital share depends on the other asset's current price since this affects the probability of the payoff event.

To employ event approximation, we must be able to value digital options and shares for a variety of payoff events. The value of a digital option is simply the discounted value of the risk-neutral probability of the payoff event Ω'

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[3]See Ingersoll [2000] for a development of the digital option and digital shares.

$$\mathcal{D}(S_1, S_2, t; T; \Omega') = e^{-r(T-t)}\hat{\text{Prob}}\{\Omega' \mid S_{1t} = S_1,\ S_{2t} = S_2\} . \tag{6}$$

As shown in Ingersoll [2000], the digital share values can be similarly related to the risk-neutral probabilities of the event Ω' in the economies using each of the underlying assets as the numeraire (since the asset price appearing in the integration is equal to one in its own numeraire).

We assume the standard Black-Scholes case of lognormal diffusions $dS_i = (\mu_i - q_i)dt + \sigma_i d\omega_i$ with μ_i, q_i and σ_i as the expected rate of return, the dividend yield for a continuously paid dividend, and the logarithmic standard deviation. The correlation between the two assets is $\rho = E[d\omega_1 d\omega_2]/dt$ with covariance $\sigma_{12} = \rho\sigma_1\sigma_2$. Then by a simple change of measure the digital share values in the dollar numeraire are

$$\mathcal{S}_k(S_1, S_2, t; T; \Omega') = S_k e^{(r-q_k)(T-t)} \mathcal{D}\left(S_1 e^{\sigma_{1k}(T-t)}, S_2 e^{\sigma_{2k}(T-t)}, t; T; \Omega'\right) . \tag{7}$$

The only remaining task is to pick the approximating event, Ω'. For European-style contracts with path-independent payoffs, it is convenient to choose for Ω' the union of sub-events $\{S_{1T} > K \cap x < S_{2T} < X\}$. They can easily be combined to approximate virtually any payoff event and the probabilities of such events are known for jointly distributed lognormal variables. The digital values for these events are[4]

$$\begin{aligned} &\mathcal{D}(S_1, S_2, t; T; K < S_{1T} \cap x < S_{2T} < X) \\ &\quad = e^{-r(T-t)}\left[\Phi_2\left(h_1(K), h_2(x), \rho\right) - \Phi_2\left(h_1(K), h_2(X), \rho\right)\right] \end{aligned} \tag{8a}$$

$$\begin{aligned} &\mathcal{S}_k(S_i, S_j, t; T; K < S_{1T} \cap x < S_{2T} < X) \\ &\quad = S_k e^{-q_k(T-t)}\left[\Phi_2\left(h_1^k(K), h_2^k(x), \rho\right) - \Phi_2\left(h_1^k(K), h_2^k(X), \rho\right)\right] \end{aligned} \tag{8b}$$

$$h_i(z) \equiv \frac{\ell n(S_i/z) + \left(r - q_i - \frac{1}{2}\sigma_i^2\right)(T-t)}{\sigma_i\sqrt{T-t}} \qquad h_i^k(z) \equiv \frac{\ell n(S_i/z) + \left(r - q_i + \sigma_{ik} - \frac{1}{2}\sigma_i^2\right)(T-t)}{\sigma_i\sqrt{T-t}}$$

and Φ_2 is the standard cumulative bivariate normal distribution.

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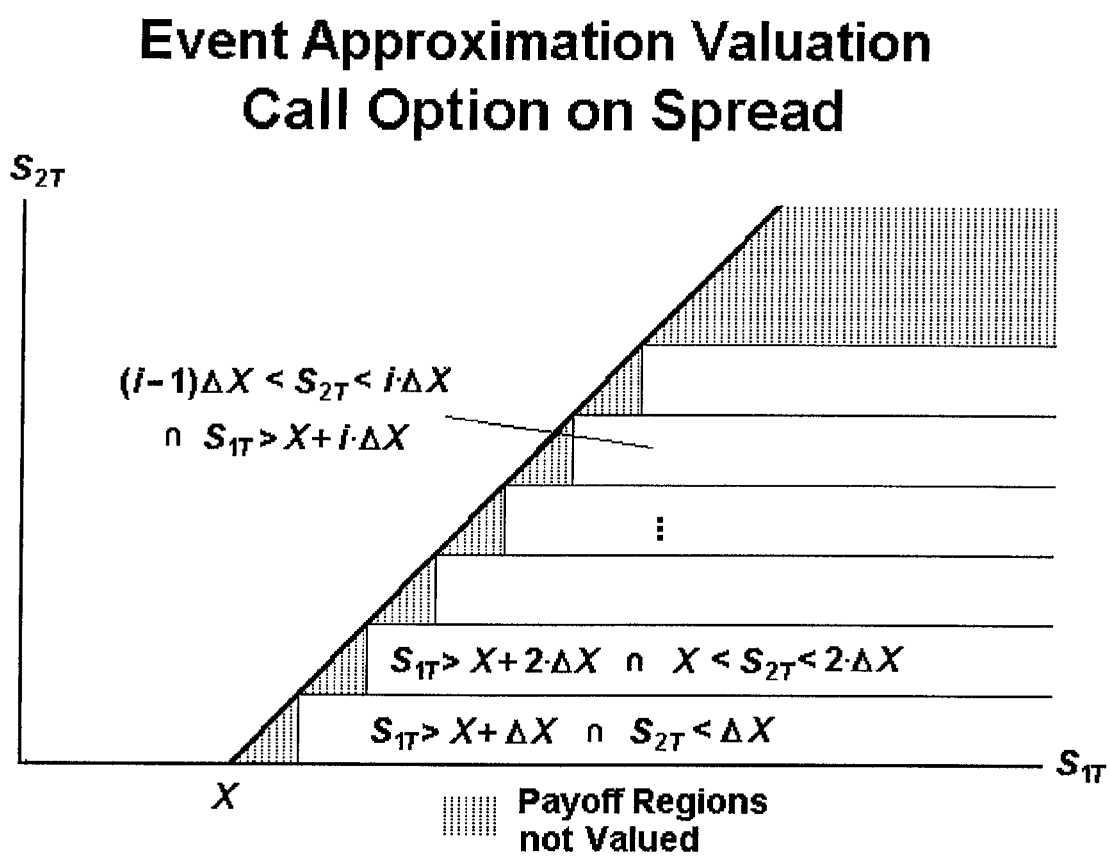
[4]Digital values for the doubly-restricted events can be determined as

$$\begin{aligned} &\mathcal{D}\left(S_1, S_2, t;\ T;\ k < S_{1T} < K \cap x < S_{2T} < X\right) \\ &\quad = \mathcal{D}\left(S_1, S_2, t; T; k < S_{1T} \cap x < S_{2T} < X\right) - \mathcal{S}\left(S_1, S_2, t; T; K < S_{1T} \cap x < S_{2T} < X\right) \end{aligned}$$

and similarly for digital shares.



As shown in Figure 1, the call on the portfolio can be approximately valued as

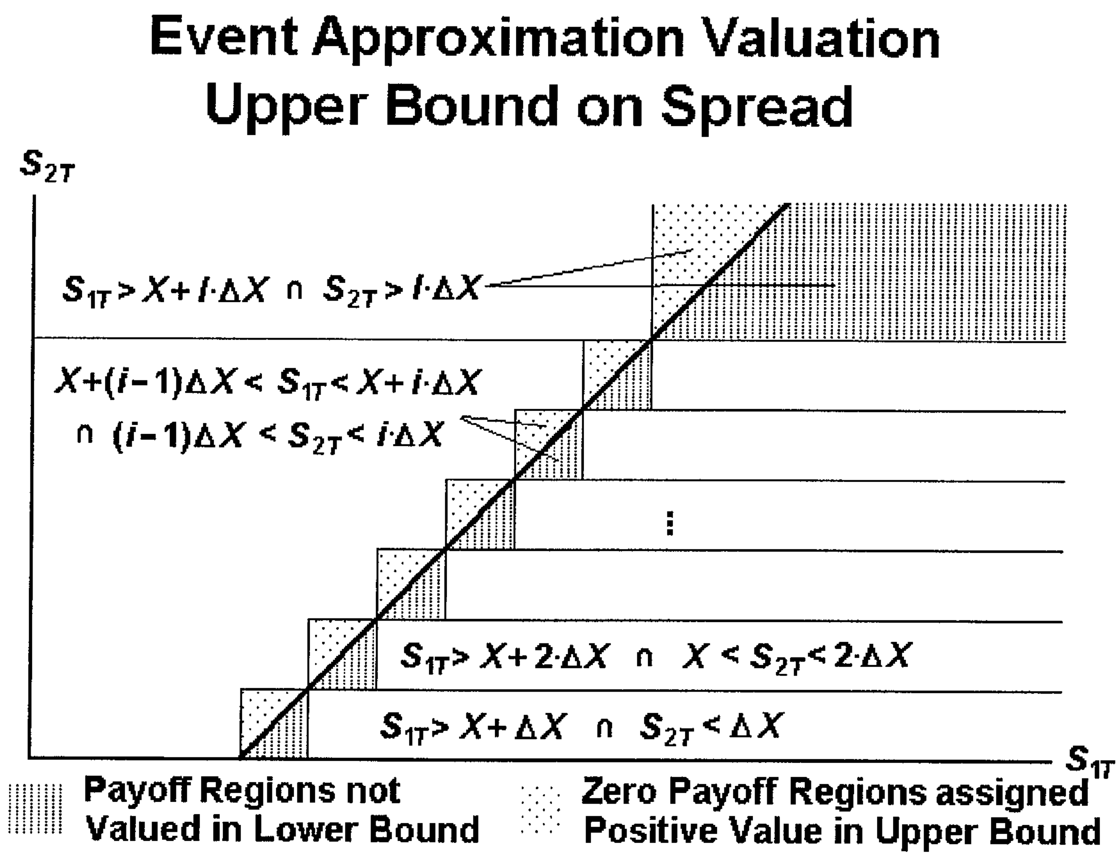
$$\text{Spread Call} \ \gtrsim \ \sum_{i=1}^{I} \mathcal{S}_1(S_1,\ S_2,\ t;\ T;\ \mathcal{E}_i) \ - \ \mathcal{S}_2(S_1,\ S_2,\ t;\ T;\ \mathcal{E}_i) \ - \ X\mathcal{D}(S_1,\ S_2,\ t;\ T;\ \mathcal{E}_i)$$

$$\text{where} \qquad \mathcal{E}_i \ \equiv \ \left\{S_{1T} > X + i\cdot\Delta X \ \cap \ (i-1)\Delta X < S_{2T} < i\cdot\Delta X\right\} \ . \qquad (9)$$

This approximation excludes each of the small shaded triangles and the larger unbounded "triangular" region above the line $S_{2T} = I\cdot\Delta X$. The payoff on the spread call in each of these excluded regions is positive; therefore, the approximation provides a lower bound to the value of the option as indicated in (9). For a sufficiently small value of ΔX and a sufficiently large value of $I\cdot\Delta X$, this approximation can be made as accurate as desired.

An upper bound for the spread option can be determined similarly. The lower bound in (9) falls short of the correct value because it misses the payoffs in the small shaded triangles and the unbounded top region of the figure. Our upper bound provides an overestimate of the value of the payoff in those regions.

The payoff in each of the shaded squares is zero in the top left (lightly shaded) portion.



In the lower right (heavily shaded) portion of each square, the payoff is $S_{1T} - S_{2T} - X$. In each of the squares $(i - 1)\Delta X \leq S_{2T}$ and $S_{1T} \leq X + i \cdot \Delta X$, so the payoff is no greater than ΔX.

In the top shaded region, S_{2T} is at least $I \cdot \Delta X$ so the payoff is no less than $S_{1T} - I \cdot \Delta X - X$. The error in the lower bound (9) is therefore less than

$$\mathcal{S}_1\left(S_1, S_2, t; T; \mathcal{E}^e\right) - (X + I \cdot \Delta X)\mathcal{D}\left(S_1, S_2, t; T; \mathcal{E}^e\right)\Big) + \Delta X \cdot \sum_{i=1}^{I} \mathcal{D}\left(S_1, S_2, t; T; \mathcal{E}^e_i\right) \tag{10}$$

$$\text{where} \quad \mathcal{E}^e \equiv S_{1T} > X + I \cdot \Delta X \cap S_{2T} > I \cdot \Delta X$$

$$\mathcal{E}^e_i \equiv X + (i-1)\Delta X < S_{1T} < X + i \cdot \Delta X \cap (i - 1)\Delta X < S_{2T} < i \cdot \Delta X \quad .$$

Rather than bounding the spread option's value by excluding regions or including extra regions, a point estimate can be computed by approximating the value of the heavily shaded triangles. If the risk-neutral probability of every point in a given shaded square were equal, the exact contribution to the payoff from each square would equal the average payoff value of that square multiplied by the risk-neutral probability of the whole square and the discount factor. Of course the risk-neutral probability is not constant throughout each square, but if the squares are small enough, the risk-neutral probabilities should be close to constant.

We know the risk-neutral probability of the each square so all we need to determine is the average payoff in the square. As noted previously the payoff in each square ranges from zero left of the slanted line to ΔX at the lower right hand corner. The equally weighted average payoff in each square is $\Delta X/6$ which is more than the zero value used in the lower bound but substantially less than the ΔX value used in the upper bound.[5] So our point estimate for the value of the spread option is

$$\text{Call Spread} \approx \sum_{i=1}^{I}\Big[\mathcal{S}_1(S_1, S_2, t; T; \mathcal{E}_i) - \mathcal{S}_2(S_1, S_2, t; T; \mathcal{E}_i) - X\cdot\mathcal{D}(S_1, S_2, t; T; \mathcal{E}_i) + \frac{\Delta X}{6}\mathcal{D}\big(S_1, S_2, t; T; \mathcal{E}_i^e\big)\Big] \tag{11}$$

where $\mathcal{E}_i$ and $\mathcal{E}_i^e$ are as defined in (9) and (10). This is probably an underestimate of the value because a portion of region above $S_{2T} = I\cdot\Delta X$ has not be valued.

As an example of using this method we price a spread option for the following scenario. The option has a maturity of one year and a strike price of $X = 10$. The two assets have prices of $S_1 = 60$ and $S_2 = 50$, and volatilities of $\sigma_1 = 40\%$ and $\sigma_2 = 30\%$. They pay no dividends. The correlation coefficient is $\rho = 0.3$, and the interest rate is $r = 6\%$. The correct value of this spread option is 9.849. Table III shows a number of different approximations along with upper and lower bounds.

As shown in the table, the spread option can be priced accurately to a penny with digitals for only 15 to 20 regions.[6] This is a much quicker and simpler procedure than solving the partial differential equation numerically as in Boyle, Evnine, and Gibbs [1989] or using numerical integration as proposed by Pearson [1995]. Even ensuring complete accuracy with a very tight upper and lower bound requires valuing only 100 regions.

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[5]Each shaded square has an area of ΔX^2. The average payoff in each is

$$\frac{1}{\Delta X^2}\int_0^{\Delta X}\int_0^{S_1}(S_1 - S_2)dS_2 dS_1 = \frac{\Delta X}{6} .$$

[6]The number of regions that need to be valued can be reduced while maintaining the same accuracy by making the "rectangles" different sizes. Since the probability of very low values of S_{2T} is small, the approximation is improved if we concentrate on valuing more likely (higher S_{2T}) outcomes. If we make the low S_{2T} outcome "rectangles" larger it increases the area of positive payoffs which are not valued and, therefore, decrease the lower bound. However, since the probability of very low values of S_{2T} is small, these extra omissions do not contribute a substantial error.

Table I: Valuation of Spread Option by Event Approximation

Point Estimate and Upper and Lower Bounds for Call on Spread

$S_1 = 60$ $S_2 = 50$ $X = 10$ $r = 6\%$
$\sigma_1 = 40\%$ $\sigma_2 = 30\%$ $\rho = 0.3$

	ΔX	N	Estimate Eq. (11)	Lower Bound Eq. (9)	Upper Bound Eq. (9) + Eq. (10)
	20.0	4	10.025	9.062	14.843
	10.0	8	9.871	9.619	11.136
$N\Delta X = 80$	5.0	16	9.851	9.788	10.177
	2.0	40	9.849	9.839	9.907
	1.0	80	9.849	9.846	9.869
	20.0	5	10.021	9.034	14.957
	10.0	10	9.870	9.612	11.162
$N\Delta X = 100$	5.0	20	9.844	9.834	9.897
	2.0	50	9.849	9.839	9.902
	1.0	100	9.849	9.846	9.863

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