YALE INTERNATIONAL CENTER FOR FINANCE

OPTIMAL DYNAMIC TRADING STRATEGIES WITH RISK LIMITS

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Abstract

Value at Risk (VaR) has emerged in recent years as a standard tool to measure and control the risk of trading portfolios. Yet, existing theoretical analyses of the optimal behavior of a trader subject to VaR limits have produced a negative view of VaR as a risk-control tool. In particular, VaR limits have been found to induce increased risk exposure in some states and an increased probability of extreme losses. However, these conclusions are based on models that are either static or dynamically inconsistent. In this paper we formulate a dynamically consistent model of optimal portfolio choice subject to VaR limits and show that the conclusions of earlier papers are incorrect if, consistently with common practice, the VaR is reevaluated dynamically making full use of conditioning information. In particular, we find that the risk exposure of a trader subject to a VaR limit is always lower than that of an unconstrained trader and that the probability of extreme losses is also lower. We also consider the Tail Conditional Expectation (TCE), a coherent risk measure often advocated as an alternative to VaR, and show that in our dynamic setting it is always possible to transform a TCE limit into an equivalent VaR limit, and conversely.

Journal of Economic Literature Classification Numbers: D91, D92, G11, C61.

1. Introduction

Investment firms customarily limit the discretionality of their traders by imposing limits on the risk of trading portfolios. Since Value at Risk (VaR) has gained in recent years increasing popularity as a risk measure,[1] these limits are frequently specified in terms of VaR.[2]

The popularity of VaR is due at least in part to the fact that it is an easily-understood measure of risk: specifically, VaR is the maximum loss of a portfolio over a given horizon, at a given confidence level. The choice of a horizon and confidence level are largely arbitrary, although the Basle Committee proposals of April 1995 prescribed that VaR computations for the purpose of assessing bank capital requirements should be based on a uniform horizon of 10 trading days (two calendar weeks) and a 99% confidence level.[3] The use of VaR as a risk measure has been endorsed by regulators and industry groups, including the Basle Committee on Banking Supervision, the SEC, the Group of Thirty (an international consultative group of leading bankers, financiers and academics), the International Swap and Derivatives Association (which represents more than 150 leading financial institutions dealing in privately negotiated over-the-counter derivatives transactions) and the Derivatives Policy Group (which comprises the six U.S. brokers-dealers with the largest OTC derivatives affiliates).[4] Both J.P. Morgan and Bankers Trust have introduced risk management systems (called RiskMetrics and RAROC 2020, respectively) that produce VaR measures.

In spite of its widespread acceptance, VaR is also known to possess unappealing features. Artzner, Delbaen, Eber and Heath (1999) proposed an axiomatic foundation of risk measures, by identifying four properties that a reasonable risk measure should satisfy and providing a characterization of the risk measures satisfying these properties, which they called *coherent risk measures*. VaR is not a coherent risk measure, as it does not satisfy the subadditivity property: in other words, the VaR associated with a combination of two portfolios can be higher than the sum of the VaRs of the two individual portfolios. This has induced Artzner, Delbaen, Eber and Heath to propose the use of the Tail Conditional expectation (TCE), defined as the conditional expectation of losses above the VaR, as an

[1] See “The Risk Business”, *The Economist*, October 17, 1998.

[2] As noted by Jorion (2001, p. 379), “At the business area or unit level, VaR [. . .] can be used to set position limits for traders and to decide where to allocate limited capital resources. A great advantage of VaR is that it creates a common denominator with which to compare various risky activities. Traditionally, position limits are set in terms of notional exposure. A trader, for instance, may have a limit of $10 million on overnight positions in 5-year Treasuries. The same limit for 30-year Treasuries or in Treasury bond futures, however, is substantially riskier. Thus, notional position limits are not directly comparable across units. Instead, VaR provides a common denominator to compare various asset classes and can be used as a guide to set position limits for business units.” As it will become clear in the following anlysis, the risk limits we consider translate naturally into position limits that take into account both the risk and the expected return of the position (see Remarks 2 and 3).

[3] Banks using a 1-day horizon for internal VaR reporting are allowed to obtain their 10-day VAR by simply multiplying the 1-day VaR by the square root of 10: see Jorion (2001, pp. 64–65).

[4] See Jorion (2001, pp. 43–49).

alternative to VaR.[5] Artzner, Delbaen, Eber and Heath proved that TCE is a coherent risk measure under a technical condition on the risk distribution. Pflug (2000) provides a general proof of coherence and discusses several additional desirable properties of TCE (see also Embrechts (1999)).

Our focus in this paper is on the dynamic portfolio choice of a trader subject to a risk limit specified in terms of VaR or TCE. This problem has not yet received a complete treatment in the existing literature.[6] Ahn, Boudoukh, Richardson and Whitelaw (1999) study the static minimization of the VaR of a given stock exposure using put options on the stock. Alexander and Baptista (2000), Huisman, Koedijk and Pownall (1999), Kast, Luciano and Peccati (1999) and Vorst (2001), among others, focus on the maximization of the expected return of a portfolio subject to a VaR constraint in a static (one-period) setting, while Rockafellar and Uryasev (2001) consider TCE minimization, again in a static setting. To our knowledge, the only analyses of portfolio choice subject to risk limits in models with dynamic trading are in two recent papers by Emmer, Klüppelberg and Korn (2001) and Basak and Shapiro (2001).[7]

Emmer, Klüppelberg and Korn consider a model with continuous trading in which traders face a VaR limit. However, for analytical tractability, they only consider strategies that maintain fixed portfolio weights: this reduces their problem to a static one and results in a dynamically-inconsistent trading strategy.

Basak and Shapiro consider the following static optimization problem:

$$\begin{aligned} &\max_{W_T \geq 0} \mathrm{E}[u(W_T)] \\ \text{s.t.} \quad &\mathrm{E}[\xi_T W_T] \leq W_0, \\ &P[W_0 - W_T > \overline{VaR}] \leq \alpha, \end{aligned} \tag{1}$$

where u is the trader's utility function, $T > 0$ is the investment horizon (which is assumed to coincide with the VaR horizon), W_T (respectively, $W_0 > 0$) is the terminal (respectively, the initial) portfolio value, ξ_T is the state-price density at time T, $1-\alpha \in (0,1)$ is the chosen confidence level and $\overline{VaR} \geq 0$. The first constraint in (1) is the usual budget constraint, while the second constraint is equivalent to the portfolio VaR being no larger than $\overline{VaR}$. The problem in (1) is interpreted as the static formulation of a dynamic portfolio problem subject to a VaR constraint at time 0 in a complete-market economy with continuous trading. Letting $\overline{\xi}$ be such that $P[\xi_T > \overline{\xi}] = \alpha$, Basak and Shapiro show that, whenever the constraint is binding, a trader forced to reduce portfolio losses in some states to satisfy the VaR constraint would optimally choose to finance these reduced losses by increasing the portfolio losses in the "costly states" where $\xi_T > \overline{\xi}$. Since these states are already the ones with the largest losses under the unconstrained optimal policy, the VaR constraint results in a fattening of the left tail of the distribution of the terminal portfolio value (i.e., in an increased probability of extreme losses). This leads Basak and Shapiro to conclude:

> *The [VaR risk-management] is viewed by many as a tool to shield economic agents from large losses, which, when they occur, could cause credit and solvency*

[5] TCE is also sometimes referred to as Conditional VaR, Tail VaR, Mean Excess Loss, Conditional Loss or Tail Loss.

[6] For a review of the literature on VaR and related risk measures, see the book of Jorion (2001), the review article of Duffie and Pan (1997) and the extensive on-line references at `www.gloriamundi.org`.

[7] Basak and Shapiro (2001) also contains an analysis of the general equilibrium implications of VaR limits.

problems. But our solution reveals that when a large loss occurs, it is a yet larger loss under the [VaR risk-management] and hence more likely to lead to credit problems, defeating the very purpose of using the [VaR risk-management]. (p. 378)[8]

Not surprisingly, Basak and Shapiro also find that, with lognormally-distributed returns, the VaR constraint in problem (1) induces traders to invest significantly more in risky assets in some states than they would have invested in the absence of the constraint: this increase in risk-taking is necessary to realize increased losses in the "costly states". Finally, Basak and Shapiro report that a risk limit specified in terms of tail-expectation-based measure would result in neither an increased probability of extreme losses nor in an increased allocation to risky assets in some states: thus, tail-expectation-based measures should be preferred to quantile-based measures (such as VaR) for the purpose of risk control.[9]

However, the problem in (1) has several shortcomings as a model of the dynamic portfolio choice of a trader subject to risk limits. First, it assumes that the portfolio's VaR is never reevaluated after the initial date: thus, the probability of portfolio losses below the prescribed maximum VaR can become zero after the initial date and yet the trader is allowed to continue to follow his trading strategy. This assumption is extreme: in practice, most financial institutions using VaR for internal risk control reevaluate it at least daily.[10] Second, because the VaR limit is only imposed at the initial date, the trading strategy solving (1) is dynamically inconsistent and must be interpreted as a commitment solution: otherwise, the trader would find it optimal to instantaneously revert to the unconstrained-optimal investment strategy after the initial date and the VaR constraint would never be binding. Third, the formulation in (1) assumes that the portfolio VaR is computed under full knowledge of the trader's behavior in all future contingencies. Again, this assumption is extreme and it does not match actual practice: as noted by Jorion (2001, p. 107), VaR is invariably computed under the assumption that the existing portfolio is kept unchanged over the VaR horizon.[11]

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[8]Similarly, Vorst (2001) states: "*Recently, financial institutions discovered that portfolios with a limited Value at Risk often showed returns that were close to the VaR and had large losses in the exceptional cases where losses exceeded VaR. [The] theoretically optimal portfolios indeed have the properties as experienced by financial institutions and illustrate that maximizing under a VaR-constraint is very dangerous.*"

[9]To reach this conclusion, Basak and Shapiro consider a tail-expectation-based risk measure, which they call *Limited-Expected-Losses* (LEL), that is computed under the equivalent martingale measure rather than under the actual probability measure.

[10]The Basle Committee proposals of April 1995 require banks to recompute the VaR of their portfolios on a daily basis, as capital requirements are proportional to the higher of the previous day's VaR or the average VaR over the last 60 business days: see Jorion (2001, p. 64). Similarly, the 1993 "Best Practices" Recommendations from the Group of Thirty stated that "Dealers should use a consistent measure to calculate *daily* the market risk of their position, which is best mesured with a Value-at-Risk (VaR) approach." (Jorion, 2001, p. 485).

[11]Because VaR measures are computed under the assumption that the existing portfolio is kept unchanged over the VaR horizon, companies normally select a VaR horizon for internal reporting purposes taking into account the turnover of the trading portfolio. Accordingly, "Commercial banks currently report their trading VaR over a daily horizon because of the liquidity and rapid turnover in their portfolios. In contrast, investment portfolios such as pension funds generally invest in less liquid assets and adjust their risk exposures only slowly, which is why a 1-month horizon is generally chosen for investment purposes." (Jorion, 2001, p. 117).

Our first contribution in this paper is to develop a more realistic dynamically-consistent model of the optimal behavior of a trader subject to risk constraints. Differently from Basak and Shapiro, we assume that the risk of the trading portfolio is reevaluated dynamically (in fact, continuously), making full use of conditioning information: thus, the trader must satisfy the specified risk limit *at all times*, rather than only at the initial date. In addition, we make the risk computations in our model consistent with practice by assuming that, when assessing the risk of a portfolio, the distribution of the portfolio value at the chosen horizon is computed assuming that the current portfolio composition is kept unchanged over this horizon. For technical convenience, we restrict ourselves to the case of lognormally-distributed returns.

When risk is measured by VaR, the optimal trading behavior under the dynamic risk limit described above is significantly different from that implied by the static VaR constraint of Basak and Shapiro and results in a much more favorable assessment of VaR as a risk-control tool. In particular, we find that the proportional allocation to risky assets is always lower than what it would have been in the absence of VaR the constraint and that the probability of extreme losses is always no larger than what it would have been in the absence of the constraint. As in Basak and Shapiro, we find that the optimal investment strategy still displays two-fund separation, the two funds being the riskless asset and the instantaneous mean-variance efficient portfolio. Thus, a dynamically-reevaluated VaR constraint does not distort the composition of the optimal portfolio of risky assets: instead, it simply impacts the relative allocation to the riskless and the risky fund.

We also consider the optimal behavior of a trader subject to a TCE limit and prove that in our setting TCE and VaR are equivalent as risk-control tools: specifically, given a dynamic TCE limit, it is always possible to identify a dynamic VaR limit that would induce the same optimal trading strategy (irrespective of the trader's preferences), and conversely. This is true in spite of the fact that TCE is a coherent risk measure, while VaR is not, and results from the fact that VaR, while not being subadditive, is comonotone additive (in the sense of Pflug (2000)):[12] in our setting, the comonotonicity property arises naturally for optimal portfolios as a result of two-fund separation.

The rest of the paper is organized as follows. Section 2 describes our model. Section 3 contains the main characterization result of optimal trading strategies under VaR constraints. Section 4 provides some explicit examples with CRRA utilities. Section 5 considers the case of TCE constraints and establishes the equivalence result. Section 6 concludes and an Appendix contains all the proofs.

2. The Model

We consider a continuous-time stochastic economy on the finite horizon $[0, T]$. Uncertainty is represented by a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, P)$, where $\mathbf{F} = \{\mathcal{F}_t\}$ is the natural filtration generated by a d-dimensional Brownian motion w.

The investment opportunities are represented by $n+1$ long-lived securities. The first security (the "bond") is a money market account earning a constant continuously-compounded

[12]Two random variables X and Y on the same probability space $(\Omega, \mathcal{F}, P)$ are said to be *comonotone* if $(X(\omega_1) - X(\omega_2))(Y(\omega_1) - Y(\omega_2)) \geq 0$ a.s. for all $\omega_1, \omega_2 \in \Omega$.

interest rate $r > 0$. The other n assets (the "stocks") are risky and their price process S (inclusive of reinvested dividends) is a n-dimensional geometric Brownian motion with drift vector $r\bar{1} + \mu$ and diffusion matrix σ, i.e.,

$$S_t = S_0 + \int_0^t I_s^S (r\bar{1} + \mu)\, ds + \int_0^t I_s^S \sigma\, dw_s,$$

where I_t^S denotes the $n \times n$ diagonal matrix with elements S_t and $\bar{1} = (1, \ldots, 1)^\top$. We assume without loss of generality that $1 \leq n \leq d$ and that $\text{rank}(\sigma) = n$.[13] Trading in the bond and in the stocks takes place continuously and is frictionless. An admissible trading strategy is an adapted n-dimensional portfolio-weight process π with $\int_0^T |\pi_t|^2\, ds < \infty$.[14] Let Π denote the set of admissible trading strategies. Given a trading strategy $\pi \in \Pi$, the associated portfolio value process W^π satisfies the dynamic budget constraint

$$W_t^\pi = W_0 + \int_0^t W_s^\pi \left(r + \pi_s^\top \mu\right) ds + \int_0^t W_s^\pi \pi_s^\top \sigma\, dw_s,$$

or

$$W_t^\pi = W_0 \exp\left(\int_0^t \left(r + \pi_s^\top \mu - \frac{1}{2} |\pi_s^\top \sigma|^2 \right) ds + \int_0^t \pi_s^\top \sigma\, dw_s \right), \tag{2}$$

where $W_0 > 0$ denotes the initial value of the portfolio. Notice that (2) implies

$$W_{t+\tau}^\pi = W_t^\pi \exp\left(\int_t^{t+\tau} \left(r + \pi_s^\top \mu - \frac{1}{2} |\pi_s^\top \sigma|^2 \right) ds + \int_t^{t+\tau} \pi_s^\top \sigma\, dw_s \right) \tag{3}$$

for any $\tau > 0$.

For given $\tau > 0$, $W > 0$ and $\pi \in \mathbb{R}^n$, let

$$\mathcal{W}_{t+\tau}(W, \pi) = W \exp\left(\left(r + \pi^\top \mu - \frac{1}{2} |\pi^\top \sigma|^2 \right) \tau + \pi^\top \sigma (w_{t+\tau} - w_t) \right).$$

It follows immediately from (3) that, given a portfolio π_t and the associated portfolio value W_t^π at time t, the random variable $\mathcal{W}_{t+\tau}(W_t^\pi, \pi_t)$ would be the future value of the portfolio at time $t + \tau$ *if the portfolio weights were kept constant between time t and time $t + \tau$.*

For a given probability level $\alpha \in (0, 1)$ and a given horizon $\tau > 0$, the VaR at time t of a portfolio $\pi \in \Pi$, denoted by $VaR_t^{\alpha,\pi}$, is then given by

$$VaR_t^{\alpha,\pi} = \inf \left\{ L \geq 0 : P\left(W_t^\pi - \mathcal{W}_{t+\tau}(W_t^\pi, \pi_t) \geq L \mid \mathcal{F}_t \right) < \alpha \right\} = (Q_t^{\alpha,\pi})^-, \tag{4}$$

where

$$Q_t^{\alpha,\pi} = \sup \left\{ L \in \mathbb{R} : P\left(\mathcal{W}_{t+\tau}(W_t^\pi, \pi_t) - W_t^\pi \leq L \mid \mathcal{F}_t \right) < \alpha \right\}$$

is the quantile of the projected portfolio gain over the interval $(t, t+\tau)$ and $x^- = \max[0, -x]$. In other words, $VaR_t^{\alpha,\pi}$ *is the loss over the next period of length τ which would be exceeded only with a (small) conditional probability α if the current portfolio π_t were kept unchanged.* The fact that $VaR_t^{\alpha,\pi}$ is computed under the assumption that the current portfolio is kept unchanged reflects the actual practice and the fact that financial institutions monitoring

[13] If $n > d$ or $\text{rank}(\sigma) < n$, some stocks are redundant and can be omitted from the analysis.

[14] All the inequalities involving random variables are understood to hold almost surely.

their traders do not typically know the traders' future portfolio choices over the VaR horizon. Instead, the measure of VaR in (4) only requires knowledge of the current portfolio value, the current portfolio composition and the conditional distribution of asset returns.[15]

Similarly, the TCE of a portfolio $\pi \in \Pi$ is defined by

$$TCE_t^{\alpha,\pi} = \left(\frac{\mathrm{E}\left[(W_t^\pi - \mathcal{W}_{t+\tau}(W_t^\pi, \pi_t)) 1_{\{W_t^\pi - \mathcal{W}_{t+\tau}(W_t^\pi, \pi_t) \geq -Q_t^{\alpha,\pi}\}} \mid \mathcal{F}_t \right]}{\alpha} \right)^+, \tag{5}$$

where $x^+ = \max[0, x]$. In other words, the TCE of a portfolio is the conditional expected value of the losses exceeding $-Q_t^{\alpha,\pi}$.

Given our assumption of lognormally-distributed asset returns, both the VaR and the TCE of a portfolio can be explicitly computed.

Proposition 1. *We have*

$$VaR_t^{\alpha,\pi} = W_t^\pi \left[1 - \exp\left(\left(r + \pi_t^\top \mu - \frac{1}{2} |\pi_t^\top \sigma|^2 \right) \tau + N^{-1}(\alpha) |\pi_t^\top \sigma| \sqrt{\tau} \right) \right]^+ \tag{6}$$

and

$$TCE_t^{\alpha,\pi} = W_t^\pi \left[1 - \exp\left((r + \pi_t^\top \mu) \tau \right) \frac{N\left(N^{-1}(\alpha) - |\pi_t^\top \sigma| \sqrt{\tau} \right)}{\alpha} \right]^+, \tag{7}$$

where $N(x)$ and $N^{-1}(x)$ denote the normal distribution and inverse distribution functions. In particular,

$$0 \leq VaR_t^{\alpha,\pi} \leq TCE_t^{\alpha,\pi} < W_t^\pi \tag{8}$$

and

$$VaR_t^{\alpha,0} = TCE_t^{\alpha,0} = 0. \tag{9}$$

Proof. See the Appendix.

3. Optimal Trading Strategies under VaR Limits

Now consider the problem of a trader who starts with an endowment W_0 and must select a portfolio $\pi \in \Pi$ so as to maximize the expected utility $\mathrm{E}[u(W_T^\pi)]$ of the terminal value of the trading portfolio, subject to the constraint that, at any time $t \in [0, T]$, the value at risk of its portfolio, $VaR_t^{\alpha,\pi}$, is no larger than some prespecified level $\overline{VaR}(W_t^\pi, t) \geq 0$:

$$\begin{aligned} & \max_{\pi \in \Pi} \mathrm{E}[u(W_T^\pi)] \\ \text{s.t.} \quad & W_0^\pi = W_0 \\ & VaR_t^{\alpha,\pi} \leq \overline{VaR}(W_t^\pi, t) \quad \forall t \in [0, T]. \end{aligned} \tag{10}$$

[15]Alternatively, it would be possible to compute the portfolio's VaR under the assumptions that the current asset holdings were kept unchanged. Our formulation not only has computational advantages under lognormality, but is also a natural one for portfolios having target compositions specified in terms of proportions rather than amounts. In addition, this formulation is consistent with typical calculations of the VaR of a portfolio, which are based on the assumption that the horizon portfolio return equals a weighted average of the horizon asset returns, with weights equals to the relative amounts invested at the beginning of the period (see Jorion (2001, Section 7.1)). In any case, the difference between the two formulations is likely to be insignificant if the VaR horizon τ is small (e.g., 1 day).

Note that in (10) we allow the VaR limit at time t to depend on calendar time and on the current value of the portfolio.

Remark 1. *Since we have assumed* $\overline{VaR}(W_t^\pi, t) \geq 0$,*it follows from (9) that setting* $\pi_t = 0$ *(that is, investing everything in the riskless bond) always satisfies the VaR constraint. Hence, the set of feasible trading strategies is not empty.*

Remark 2. *The expression for VaR in (6) implies that a portfolio* π *satisfies the constraint* $VaR_t^{\alpha,\pi} \leq \overline{VaR}(W_t^\pi, t)$ *if and only if*

$$\log\left(1 - \frac{\overline{VaR}(W_t^\pi, t)}{W_t^\pi}\right)^+ - \left(r + \pi_t^\top \mu - \frac{1}{2}|\pi_t^\top \sigma|^2\right)\tau - N^{-1}(\alpha)|\pi_t^\top \sigma|\sqrt{\tau} \leq 0. \tag{11}$$

In the case of a single risky asset ($n = 1$)*, it can be easily verified that (11) is equivalent to an upper and a lower bound on the fraction* π_t *allocated to the risky asset:*

$$\pi^-(W_t^\pi, t) \leq \pi_t \leq \pi^+(W_t^\pi, t),$$

where

$$\pi^\pm(W, t) = \frac{\frac{\mu}{|\sigma|}\sqrt{\tau} \pm N^{-1}(\alpha) \pm \sqrt{\left(\frac{\mu}{|\sigma|}\sqrt{\tau} \pm N^{-1}(\alpha)\right)^2 - 2\left(\log\left(1 - \frac{\overline{VaR}(W,t)}{W}\right)^+ - r\tau\right)}}{|\sigma|\sqrt{\tau}}. \tag{12}$$

In particular, given the current portfolio value W_t^π*, the set of admissible portfolios* π_t *is convex. This is however not necessarily the case in the presence of multiple risky assets. Figure 1 shows an example with two risky assets in which the set of admissible trading strategies is not convex.*[16]

Rewriting (10) as the stochastic control problem

$$\begin{aligned} &\max_{\pi \in \Pi} \mathrm{E}[u(W_T^\pi)] \\ \text{s.t.} \quad & W_t^\pi = W_0 + \int_0^t W_s^\pi \,(r + \pi_s^\top \mu)\, ds + \int_0^t W_s^\pi \pi_s^\top \sigma \, dw_s, \\ & \log\left(1 - \frac{\overline{VaR}(W_t^\pi, t)}{W_t^\pi}\right)^+ - \left(r + \pi_t^\top \mu - \frac{1}{2}|\pi_t^\top \sigma|^2\right)\tau - N^{-1}(\alpha)|\pi_t^\top \sigma|\sqrt{\tau} \leq 0 \end{aligned} \tag{13}$$

leads to the following characterization of optimal trading strategies.[17]

[16] Note that in this example both assets have a return risk premium of .07 and a volatility of .17, while the instantaneous correlation coefficient between the two risky assets is -.9.

[17] Since the set of π_t satisfying the inequality (11) is not necessarily convex when $n > 1$ (as shown in Figure 1) and it depends on the current portfolio value W_t^π, the convex duality technique of Cvitanić and Karatzas (1992) cannot be applied to this problem.

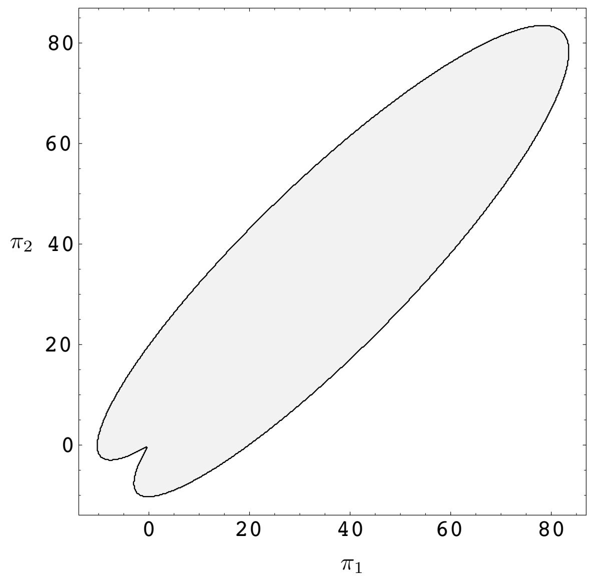

Figure 1: The graph shows the set of portfolios satisfying the VaR constraint, assuming $n = 2$, $r = .008$, $\mu = \begin{pmatrix} .07 \\ .07 \end{pmatrix}$, $\sigma = \begin{pmatrix} .170 & 0 \\ -.153 & .074 \end{pmatrix}$, $\frac{\overline{VaR}(W_t^\pi, t)}{W_t^\pi} = .01$, $\alpha = .9$, $\tau = 1$.

Theorem 1. *Let $V(W,t)$ denote the value function for the stochastic control problem (13) and let*

$$\varphi_\alpha^+(W,t) = \frac{|\kappa|\sqrt{\tau} + N^{-1}(\alpha) + \sqrt{(|\kappa|\sqrt{\tau} + N^{-1}(\alpha))^2 - 2\left(\log\left(1 - \frac{\overline{VaR}(W,t)}{W}\right)^+ - r\tau\right)}}{|\kappa|\sqrt{\tau}}, \tag{14}$$

where $\kappa = \sigma^\top(\sigma\sigma^\top)^{-1}\mu$. Then $\varphi_\alpha^+(W,t) \geq 0$ for all $(W,t) \in (0,\infty) \times [0,T]$ and V solves the Hamilton-Jacobi-Bellman equation

$$0 = \begin{cases} -\frac{1}{2}\frac{V_W^2}{V_{WW}}|\kappa|^2 + V_W W r + V_t & \text{if } -\frac{V_W}{W V_{WW}} \leq \varphi_\alpha^+ \\ \frac{1}{2}V_{WW}W^2|\kappa\varphi_\alpha^+|^2 + V_W W(r + |\kappa|^2\varphi_\alpha^+) + V_t & \text{otherwise} \end{cases} \tag{15}$$

with terminal condition

$$V(W,T) = u(W). \tag{16}$$

Finally, letting

$$\varphi(W,t) = \min\left[-\frac{V_W(W,t)}{W V_{WW}(W,t)}, \varphi_\alpha^+(W,t)\right] \tag{17}$$

the policy

$$\pi^*(W,t) = \varphi(W,t)(\sigma\sigma^\top)^{-1}\mu \tag{18}$$

solves (13).

Proof. See the Appendix.

Remark 3. *Equation (18) shows that the optimal portfolio of a VaR-constrained agent is a combination of the riskless asset and the growth-optimal portfolio* $(\sigma\sigma^\top)^{-1}\mu$.[18] *Thus, with lognormally-distributed asset returns, the VaR constraint affects the distribution of the optimal portfolio between riskless and risky assets, but does not distort the composition of the optimal portfolio of risky assets. The function* $\varphi_\alpha^+(W,t)$ *in (14) identifies the maximum fraction of wealth that can be invested in the growth-optimal portfolio at time* t *under the VaR constraint*[19] *(as shown in the proof of Theorem 1, shorting the growth-optimal portfolio is never optimal).*

The result in Theorem 1 also allows us to compute the distribution of the terminal portfolio value under the optimal trading strategy.

Corollary 1. *Let* $p(W,t)$ *denote the density function of* $W_t^{\pi^*}$. *Then* p *solves Kolmogorov's forward equation*

$$\frac{\partial}{\partial t}p = \frac{\partial^2}{\partial W^2}\left[(W\varphi|\kappa|)^2 p\right] - \frac{\partial}{\partial W}\left[W(r+\varphi|\kappa|^2)p\right](Wp) \tag{19}$$

with initial condition

$$p(W,0) = \delta(W - W_0),$$

where φ *is the function in (17) and* δ *denotes Dirac's delta function.*

Proof. See Karatzas and Shreve (1988).

4. Examples with CRRA Utility

We now specialize our model by assuming that $u(W) = \frac{W^{1-\gamma}}{1-\gamma}$ for some $\gamma > 0$. We recall that, in the absence of a VaR constraint,

$$V(W,t) = e^{\rho(T-t)}\frac{W^{1-\gamma}}{1-\gamma},$$

where

$$\rho = (1-\gamma)\left(r + \frac{|\kappa|^2}{2\gamma}\right),$$

and

$$\pi^*(W,t) = \frac{1}{\gamma}(\sigma\sigma^\top)^{-1}\mu. \tag{20}$$

Thus, it follows from (2) and (20) that the terminal portfolio value $W_T^{\pi^*}$ is in this case lognormally distributed, with mean

$$W_0 e^{\left(r+\frac{|\kappa|^2}{\gamma}\right)T}$$

[18] The growth-optimal portfolio is the portfolio that maximizes the expected continuously-compounded rate of return $\frac{1}{T}\log(W_T^\pi/W_0^\pi)$. Equivalently, π^* is a combination of the riskless asset and the mean-variance efficient portfolio of risky assets $(\sigma\sigma^\top)^{-1}\mu/\bar{1}^\top(\sigma\sigma^\top)^{-1}\mu$.

[19] This can be immediately verified by noting that the expression for φ_α^+ in (14) can be obtained from the expression for π^+ in (12) by replacing μ with $|\kappa|^2 = \mu^\top(\sigma\sigma^\top)^{-1}\mu$ (the instantaneous risk premium on the growth-optimal portfolio) and $|\sigma|$ with $|\kappa| = |\mu^\top(\sigma\sigma^\top)^{-1}\sigma|$ (the volatility of the growth-optimal portfolio).

and standard deviation

$$W_0 e^{\left(r+\frac{|\kappa|^2}{\gamma}\right)T}\sqrt{e^{\frac{|\kappa|^2}{\gamma}T}-1}.$$

To further understand the implications of VaR constraints for optimal trading strategies, we consider below three alternative specifications of the function $\overline{VaR}(W,t)$ which identifies the maximum admissible VaR at any time $t \in [0,T]$. Notice that it follows immediately from Remark 3 and (20) that a given VaR constraint not binding if and only if

$$\frac{1}{\gamma} \leq \inf_{(W,t)\in(0,\infty)\times[0,T]} \varphi_\alpha^+(W,t).$$

Moreover, it follows from (18) and (20) that the VaR-constrained optimal portfolio is a multiple

$$q(W,t) = \gamma\varphi(W,t) \tag{21}$$

of the unconstrained optimal portfolio. Following the terminology of Basak and Shapiro (2001), we will refer to this multiple as the *relative risk exposure*. In particular, we have from the boundary condition (16) that at the terminal date

$$q(W,T) = \min[1, \gamma\varphi_\alpha^+(W,T)].$$

In cases where an analytical solution is not available, we solve the PDE (15) numerically by rewriting it in terms of the state variable $w = \log(W)$ and then applying the explicit finite-difference method (see Kushner and Dupuis (1992) for details). This allows us to obtain the optimal trading strategy π^* from (18). Since the finite-difference method approximates the state variable W^{π^*} with a Markov chain with known transition probabilities, we use these transition probabilities to compute the distribution of the terminal portfolio value $W_T^{\pi^*}$. This approach generates results similar to those obtainable by solving the Kolmogorov equation (19) separately using the finite-difference method.

All the numerical computations assume $r = .008$, $|\kappa| = .37$, $\alpha = .05$ $\tau = 1$, $T = 10$, $\gamma = .5$ or $\gamma = 5$ and $W_0 = 1$. Moreover, we scale the function $\overline{VaR}(W,t)$ so that $\overline{VaR}(W_0,0) = .5$ (this amounts to setting $\beta = .5$ in the examples below).

4.1. $\overline{VaR}(W,t) = \beta$

We start by considering the case of a constant VaR limit: $\overline{VaR}(W,t) = \beta$. In this case, the function φ_α^+ is independent of t and monotonically decreasing in W, with

$$\lim_{W\downarrow 0} \varphi_\alpha^+(W,t) = +\infty$$

and

$$\lim_{W\uparrow+\infty} \varphi_\alpha^+(W,t) = \frac{|\kappa|\sqrt{\tau} + N^{-1}(\alpha) + \sqrt{(|\kappa|\sqrt{\tau} + N^{-1}(\alpha))^2 + 2r\tau}}{|\kappa|\sqrt{\tau}} = .0169.$$

Therefore, the VaR constraint is binding if $\gamma < \frac{1}{.0169} = 59.1$ and not binding otherwise.

Figure 2 plots the relative risk exposure under the optimal policy (the function $q(W,t)$ in (21)) for the case $\gamma = .5$, when $t = 0$ and when $t = T$. Contrary to the conclusion

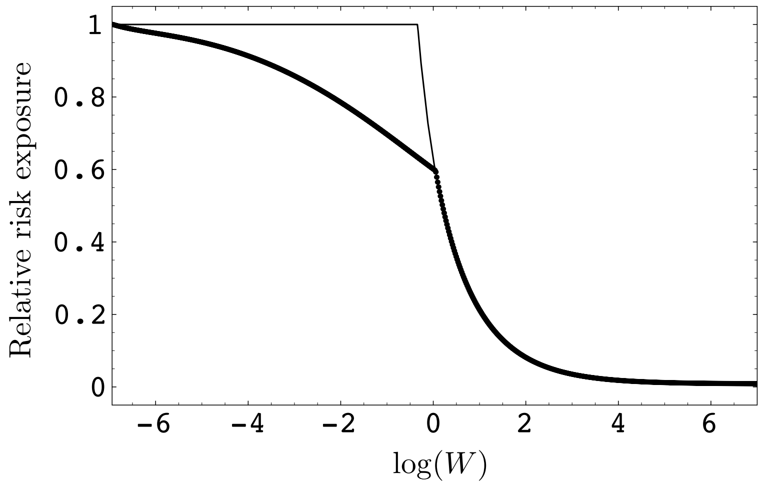

Figure 2: The graph plots the relative risk exposure $q(W,t)$ at $t = 0$ (heavier line) and $t = T$ (lighter line), assuming $r = .008$, $|\kappa| = .37$, $T = 10$, $\gamma = .5$, $W_0 = 1$, $\alpha = .05$, $\tau = 1$, $\overline{VaR}(W,t) = .5$.

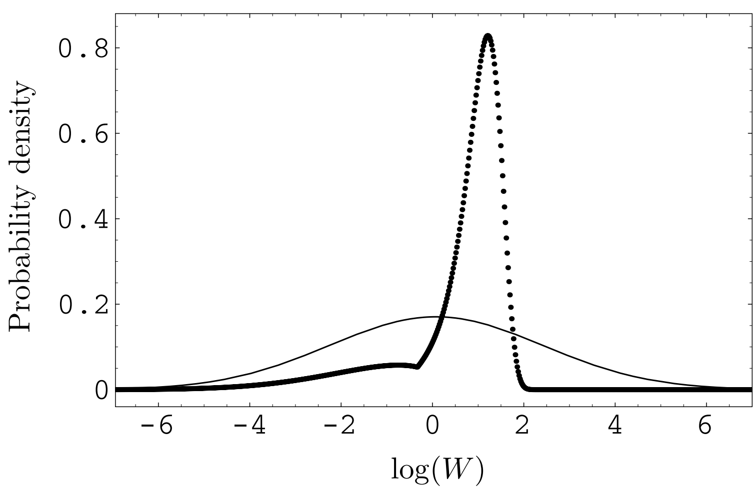

Figure 3: The graph plots the probability density of the terminal portfolio value $W_T^{\pi^*}$ under the optimal trading strategy in the constrained (heavier line) and unconstrained (lighter line) case, assuming $r = .008$, $|\kappa| = .37$, $T = 10$, $\gamma = .5$, $W_0 = 1$, $\alpha = .05$, $\tau = 1$, $\overline{VaR}(W,t) = .5$.

of Basak and Shapiro (2001), the figure shows that a VaR-constrained agent never invests more in risky assets than a VaR-unconstrained agent (the relative risk exposure is never larger than 1). Consequently, as shown in Figure 3, the probability of extreme losses at the horizon T is lower under the VaR-constrained investment strategy than under the unconstrained strategy. These conclusions also apply to the other examples we consider. Thus, the reservations expressed by Basak and Shapiro (2001), Vorst (2001) and others against the use of VaR as a risk-control tool seem unwarranted if the VaR is reevaluated periodically.

It is also worth noting the presence of a significant hedging demand in Figure 2: for example, a VaR-constrained agent with initial wealth $W_0 = .5$ ($\log(W_0) = -.693$) and an investment horizon of 10 years would invest only 66% as much as an unconstrained agent in the growth-optimal portfolio, even though he could invest any positive amount in the growth-optimal portfolio and still satisfy the VaR constraint at time 0 (since $\varphi_\alpha^+(.5, 0) = +\infty$). Clearly, this lower allocation to the growth-optimal portfolio reduces the volatility of the optimal portfolio and reflects the smaller indirect utility of extreme portfolio values induced by the fact that a constant VaR constraint becomes more severely binding when the portfolio value increases (as can be seen from Figure 2 or from the fact that the function φ^+ in (14) is a decreasing function of the portfolio value in this case).

Figures 4 and 5 illustrate the corresponding results for the case $\gamma = 5$. In this case, hedging demand is negligible, as shown by the fact that the optimal risk exposure at the initial date is very close to that at the terminal date. This stems from the fact that the VaR constraint is binding at time t (i.e.,$\frac{1}{\gamma} > \varphi_\alpha^+(W, t)$) only if $W > 5.87$ ($\log(W) > 1.77$), and this event has negligible probability, as shown in Figure 5.

4.2. $\overline{VaR}(W, t) = \beta W$

Fixing the VaR limit to a constant amount has the obvious shortcoming that the constraint becomes binding when the portfolio value increases and is not binding when the portfolio value is sufficiently low. Thus, a constant VaR limit penalizes successful traders. In practice, successful traders typically see their VaR limit increased. To capture this fact, we consider next the case of a constant proportional VaR, $\overline{VaR}(W, t) = \beta W$.

Thus,

$$\varphi_\alpha^+(W, t) = \frac{|\kappa|\sqrt{\tau} + N^{-1}(\alpha) + \sqrt{(|\kappa|\sqrt{\tau} + N^{-1}(\alpha))^2 - 2\left(\log(1-\beta)^+ - r\tau\right)}}{|\kappa|\sqrt{\tau}} = \varphi_\alpha^+ \quad (22)$$

for all (W, t). It can be easily verified that in this case the value function

$$V(W, t) = e^{\hat{\rho}(T-t)} \frac{W^{1-\gamma}}{1-\gamma},$$

where

$$\hat{\rho} = (1-\gamma)\left(r + \varphi^*\left(1 - \frac{\gamma\varphi^*}{2}\right)|\kappa|^2\right)$$

and

$$\varphi^* = \min\left(\frac{1}{\gamma}, \varphi_\alpha^+\right),$$

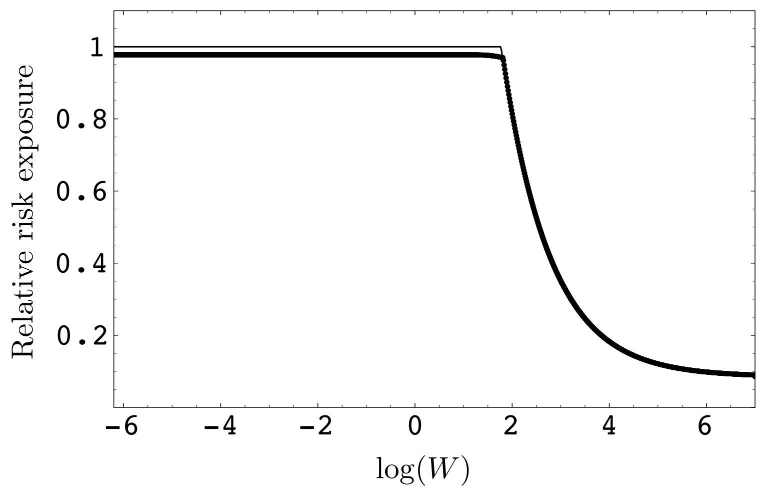

Figure 4: The graph plots the relative risk exposure $q(W,t)$ at $t=0$ (heavier line) and $t=T$ (lighter line), assuming $r=.008$, $|\kappa|=.37$, $T=10$, $\gamma=5$, $W_0=1$, $\alpha=.05$, $\tau=1$, $\overline{VaR}(W,t)=.5$.

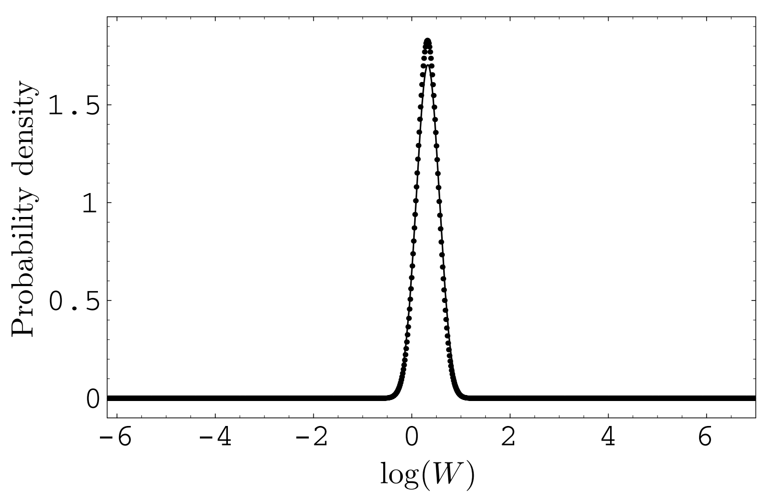

Figure 5: The graph plots the probability density of the terminal portfolio value $W_T^{\pi^*}$ under the optimal trading strategy in the constrained (heavier line) and unconstrained (lighter line) case, assuming $r=.008$, $|\kappa|=.37$, $T=10$, $\gamma=5$, $W_0=1$, $\alpha=.05$, $\tau=1$, $\overline{VaR}(W,t)=.5$.

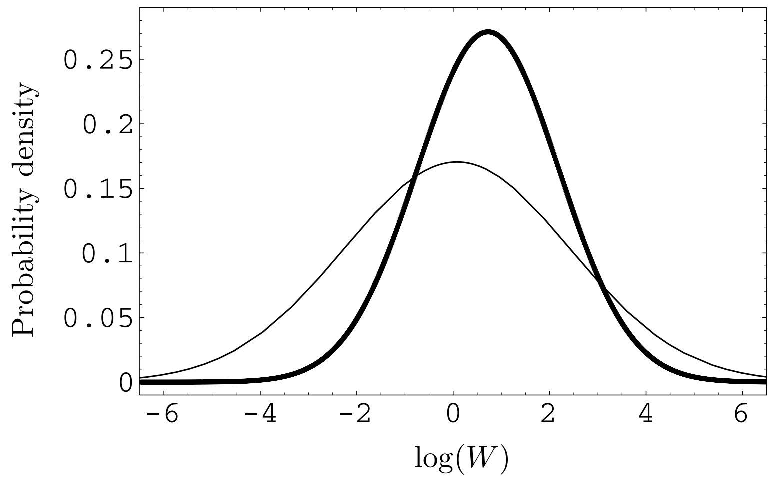

Figure 6: The graph plots the probability density of the terminal portfolio value $W_T^{\pi^*}$ under the optimal trading strategy in the constrained (heavier line) and unconstrained (lighter line) case, assuming $r = .008$, $|\kappa| = .37$, $T = 10$, $\gamma = .5$, $W_0 = 1$, $\alpha = .05$, $\tau = 1$, $\overline{VaR}(W,t) = .5W$.

solves the HJB equation (15). Thus, $\varphi(W,t) = \varphi^*$ and $q(W,t) = \gamma\varphi^* \leq 1$ for all (W,t). Hence, in this case there is no hedging demand and the VaR-constrained optimal trading strategy coincides with the unconstrained-optimal trading strategy for an investor with CRRA coefficient $\gamma^* = \frac{1}{\varphi^*}$.

With the parameters of our numerical example, $\varphi_\alpha^+ = 1.2571$, so that the VaR constraint is binding if $\gamma < \frac{1}{1.2571} = .795$ and not binding otherwise. In particular, if $\gamma = .5$ the constrained-optimal trading strategy coincides with the unconstrained trading strategy for an agent with higher CRRA coefficient $\gamma^* = .795$ and the relative risk exposure $q(W,t)$ is constant and equal to .629. Figure 6 shows the distribution of the terminal portfolio value under the VaR-constrained and the unconstrained optimal trading strategies in this case.

4.3. $\overline{VaR}(W,t) = (W - (1-\beta)W_0)^+$

In the example of Section 4.1. (constant VaR) the maximum allowable proportional investment in the growth-optimal portfolio, φ_α^+, was a decreasing function of the current portfolio value, while in the example of Section 4.2. (constant proportional VaR) it was a constant. As a final example, we consider the case in which $\overline{VaR}(W,t) = (W - (1-\beta)W_0)^+$: thus, the VaR limit equals a fixed proportion βW_0 of the initial portfolio value, plus any running gain $W - W_0$. In this case, $\varphi_\alpha^+(W,t)$ is a monotonically increasing function of W, with

$$\lim_{W\downarrow 0} \varphi_\alpha^+(W,t) = \frac{|\kappa|\sqrt{\tau} + N^{-1}(\alpha) + \sqrt{(|\kappa|\sqrt{\tau} + N^{-1}(\alpha))^2 + 2r\tau}}{|\kappa|\sqrt{\tau}} = .0169$$

and

$$\lim_{W\uparrow +\infty} \varphi_\alpha^+(W,t) = +\infty.$$

Therefore, the VaR constraint is binding if $\gamma < \frac{1}{.0169} = 59.1$ and not binding otherwise (as in the example of Section 4.1.).

It is also worth noting that, in the extreme case $\alpha = 0$, $\varphi_\alpha^+(W,t) = 0$ for all $W \leq (1-\beta)W_0$, so that the optimal portfolio π^* has the property that $W_t^{\pi^*} \geq (1-\beta)W_0$ for all $t \in [0,T]$. Thus, the dynamic VaR constraint $VaR_t^{\alpha,\pi} \leq (W_t^\pi - (1-\beta)W_0)^+$ can be considered as a relaxed version of the dynamic portfolio insurance constraint $W_t^\pi \geq (1-\beta)W_0$ for all $t \in [0,T]$.

Figures 7 and 8 show the optimal risk exposure and the distribution of the terminal portfolio value when $\gamma = .5$. In this case, the VaR constraint results in a highly skewed distribution for the terminal portfolio value, as a reduced probability of a loss larger than βW_0 in the VaR-constrained case is compensated by an increased probability of a small loss, but the probability of a large gain is virtually the same as in the unconstrained case. As already noted, when $\alpha = 0$ the resulting distribution must assign zero probability to values of $\log(W_T^{\pi^*})$ below $\log[(1-\beta)W_0] = -.693$: in Figure 8 the probability of these values is positive but negligible (less than .015, compared with a probability of .37 under the unconstrained optimal policy).

For a higher level of risk aversion, the constraint only binds at wealth levels that have negligible probability under the unconstrained optimal policy: thus, hedging demand is reduced and the constrained-optimal terminal portfolio distribution is closer to the unconstrained-optimal distribution. This is shown in Figures 9 and 10 for the case $\gamma = 5$.

5. TCE Limits

We now turn to the problem of a trader subject to a risk limit specified in terms of TCE,

$$\begin{aligned} & \max_{\pi\in\Pi} \mathrm{E}[u(W_T^\pi)] \\ \text{s.t.} \quad & W_0^\pi = W_0 \\ & TCE_t^{\hat\alpha,\pi} \leq \overline{TCE}(W_t^\pi, t) \quad \forall t \in [0,T], \end{aligned} \tag{23}$$

where $\overline{TCE}$ is a given nonnegative function and $\hat\alpha \in (0,1)$. As mentioned in the Introduction, TCE has been advocated as a better risk management tool than VaR. As we will see, however, any dynamic risk limit formulated in terms of TCE can be easily mapped into an equivalent VaR limit, and conversely, so that the choice of VaR or TCE as a risk-management tool is largely irrelevant.

Definition. The constraints $VaR_t^{\alpha,\pi} \leq \overline{VaR}(W_t^\pi, t)$ and $TCE_t^{\hat\alpha,\pi} \leq \overline{TCE}(W_t^\pi, t)$ are *equivalent* if the optimal portfolio policies in (10) and (23) coincide for all utility functions u.

Recall from Proposition 1 that

$$VaR_t^{\alpha,\pi} = W_t^\pi \rho_\alpha(\pi_t)^+$$

and

$$TCE_t^{\hat\alpha,\pi} = W_t^\pi \hat\rho_{\hat\alpha}(\pi_t)^+,$$

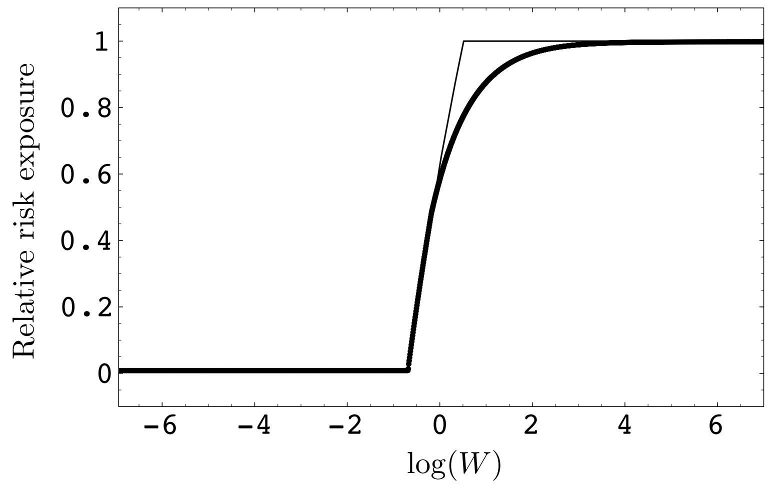

Figure 7: The graph plots the relative risk exposure $q(W,t)$ at $t = 0$ (heavier line) and $t = T$ (lighter line), assuming $r = .008$, $|\kappa| = .37$, $T = 10$, $\gamma = .5$, $W_0 = 1$, $\alpha = .05$, $\tau = 1$, $\overline{VaR}(W,t) = (W - .5W_0)^+$.

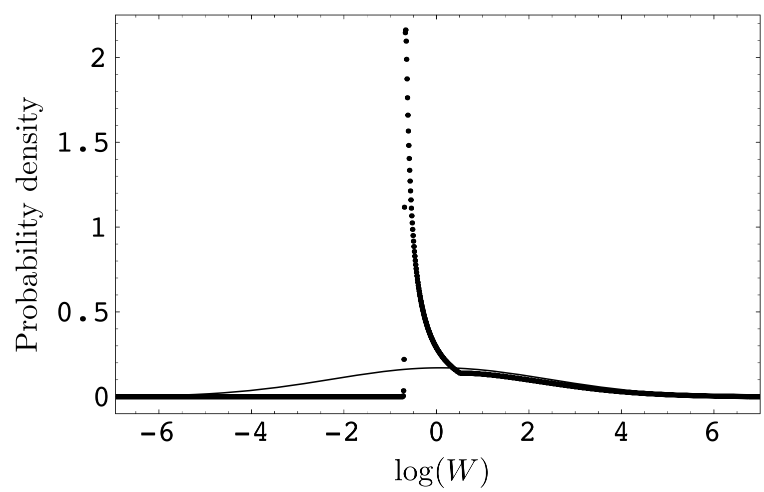

Figure 8: The graph plots the probability density of the terminal portfolio value $W_T^{\pi^*}$ under the optimal trading strategy in the constrained (heavier line) and unconstrained (lighter line) case, assuming $r = .008$, $|\kappa| = .37$, $T = 10$, $\gamma = .5$, $W_0 = 1$, $\alpha = .05$, $\tau = 1$, $\overline{VaR}(W,t) = (W - .5W_0)^+$.

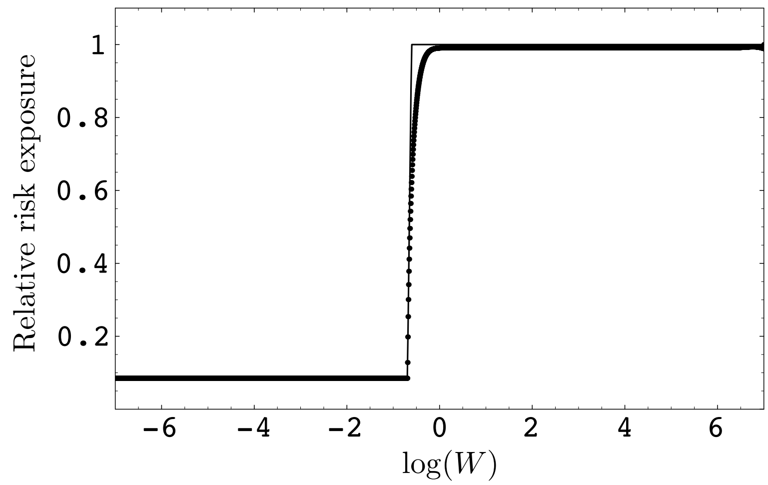

Figure 9: The graph plots the relative risk exposure $q(W,t)$ at $t = 0$ (heavier line) and $t = T$ (lighter line), assuming $r = .008$, $|\kappa| = .37$, $T = 10$, $\gamma = 5$, $W_0 = 1$, $\alpha = .05$, $\tau = 1$, $\overline{VaR}(W,t) = (W - .5W_0)^+$.

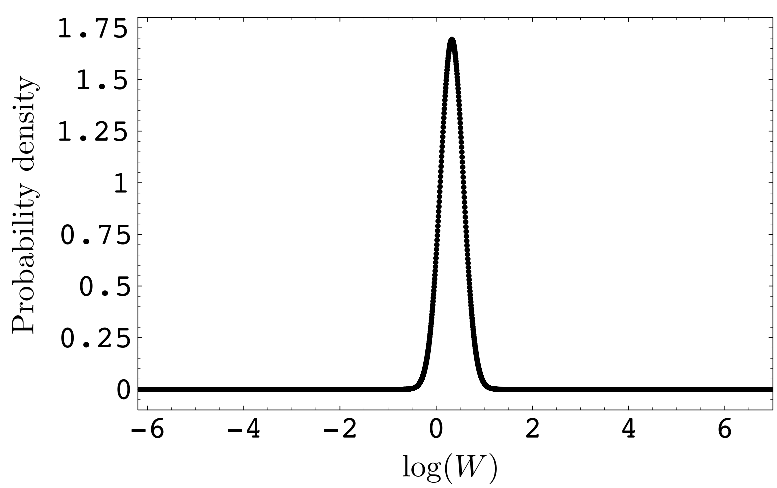

Figure 10: The graph plots the probability density of the terminal portfolio value $W_T^{\pi^*}$ under the optimal trading strategy in the constrained (heavier line) and unconstrained (lighter line) case, assuming $r = .008$, $|\kappa| = .37$, $T = 10$, $\gamma = 5$, $W_0 = 1$, $\alpha = .05$, $\tau = 1$, $\overline{VaR}(W,t) = (W - .5W_0)^+$.

where

$$\rho_\alpha(\pi_t) = 1 - \exp\left(\left(r + \pi_t^\top \mu - \frac{1}{2}|\pi_t^\top \sigma|^2\right)\tau + N^{-1}(\alpha)|\pi_t^\top \sigma|\sqrt{\tau}\right)$$

and

$$\hat{\rho}_{\hat{\alpha}}(\pi_t) = 1 - \exp\left((r + \pi_t^\top \mu)\,\tau\right) \frac{N\left(N^{-1}(\hat{\alpha}) - |\pi_t^\top \sigma|\sqrt{\tau}\right)}{\hat{\alpha}}.$$

Since $TCE_t^{\alpha,\pi}$ depends on π only through the instantaneous expected rate of return $\pi_t^\top \mu$ and the instantaneous return variance $|\pi_t^\top \sigma|^2$, it follows as in Theorem 1 that the optimal portfolio π^* in (23) is a combination of the riskless asset and the growth-optimal portfolio $(\sigma\sigma^\top)^{-1}\mu$, i.e.,

$$\pi_t^* = \hat{\varphi}(W_t^{\pi^*}, t)(\sigma\sigma^\top)^{-1}\mu \tag{24}$$

for some function $\hat{\varphi}$. The next lemma implies that, for portfolios that are combinations of the riskless asset and the growth-optimal portfolio, the TCE limit is equivalent to a lower and an upper bound on the allocation to the growth-optimal portfolio.

Lemma 1. *For all $a \geq 0$, the set*

$$\Phi_{\hat{\alpha}}^a = \left\{\varphi \in I\!R : \hat{\rho}_{\hat{\alpha}}(\varphi(\sigma\sigma^\top)^{-1}\mu) \leq a\right\}$$

is a closed interval containing the origin. Thus, there exist functions $\Phi_{\hat{\alpha}}^-$ and $\Phi_{\hat{\alpha}}^+$ with $\Phi_{\hat{\alpha}}^- < 0 < \Phi_{\hat{\alpha}}^+$ such that $\Phi_{\hat{\alpha}}^a = [\Phi_{\hat{\alpha}}^-(a), \Phi_{\hat{\alpha}}^+(a)]$. If $a < 1$, $\Phi_{\hat{\alpha}}^a$ is bounded and $\Phi_{\hat{\alpha}}^-(a)$ and $\Phi_{\hat{\alpha}}^+(a)$ are the two roots of the equation $\hat{\rho}_{\hat{\alpha}}(\varphi(\sigma\sigma^\top)^{-1}\mu) = a$. If $a \geq 1$, $\Phi_{\hat{\alpha}}^-(a) = -\infty$ and $\Phi_{\hat{\alpha}}^+(a) = +\infty$.

Proof. See the Appendix. □

Recalling that $TCE_t^{\hat{\alpha},\pi} = W_t^\pi \hat{\rho}_{\hat{\alpha}}(\pi_t)^+$, it follows immediately from the above lemma that the policy π^* in (24) satisfies the TCE limit in (23) if and only if

$$\hat{\varphi}_{\hat{\alpha}}^-(W, t) \leq \hat{\varphi}(W, t) \leq \hat{\varphi}_{\hat{\alpha}}^+(W, t)$$

for all $(W, t) \in I\!R^+ \times [0, T]$, where

$$\hat{\varphi}_{\hat{\alpha}}^\pm(W, t) = \Phi_{\hat{\alpha}}^\pm\left(\frac{\overline{TCE}(W, t)}{W}\right). \tag{25}$$

Since shorting the growth-optimal portfolio is never optimal,[20] it is then easy to show that given VaR and TCE limits are equivalent if and only if the maximum feasible allocation to the growth-optimal portfolios under the two constraints coincide, i.e., if the function φ_α^+ in (14) coincides with the function $\hat{\varphi}_{\hat{\alpha}}^+$ in (25). This leads to the following result.

Proposition 2. *The constraint $TCE_t^{\hat{\alpha},\pi} \leq \overline{TCE}(W_t^\pi, t)$ is equivalent to the constraint $VaR_t^{\alpha,\pi} \leq \overline{VaR}(W_t^\pi, t)$, where $\alpha \in (0, 1)$ is an arbitrary probability such that*

$$\frac{|\kappa|\sqrt{\tau} + N^{-1}(\alpha) + \sqrt{(|\kappa|\sqrt{\tau} + N^{-1}(\alpha))^2 + 2r\tau}}{|\kappa|\sqrt{\tau}} \leq \inf_{(W,t) \in I\!R^+ \times [0,T]} \hat{\varphi}_{\hat{\alpha}}^+(W, t), \tag{26}$$

[20] This follows from the same argument used in the proof of Theorem 1.

$$\overline{VaR}(W,t) = W\rho_\alpha(\hat{\varphi}^+_{\hat{\alpha}}(W,t)(\sigma\sigma^\top)^{-1}\mu) \geq 0, \tag{27}$$

and $\varphi^+_{\hat{\alpha}}$ *is the function in (25). Conversely, the constraint* $VaR^{\alpha,\pi}_t \leq \overline{VaR}(W^\pi_t, t)$ *is equivalent to the constraint* $TCE^{\hat{\alpha},\pi}_t \leq \overline{TCE}(W^\pi_t, t)$*, where* $\hat{\alpha} \in (0,1)$ *is an arbitrary probability such that*

$$\Phi^+_{\hat{\alpha}}(0) \leq \inf_{(W,t)\in \mathbb{R}^+\times[0,T]} \varphi^+_\alpha(W,t), \tag{28}$$

$$\overline{TCE}(W,t) = W\hat{\rho}_{\hat{\alpha}}(\varphi^+_\alpha(W,t)(\sigma\sigma^\top)^{-1}\mu) \geq 0, \tag{29}$$

and φ^+_α *is the function in (14)*

Proof. See the Appendix. □

Proposition 2 implies in particular that a proportional VaR limit is equivalent to a proportional TCE limit.

Corollary 2. *A proportional VaR limit* $VaR^{\alpha,\pi}_t \leq \beta W$ *with* $\beta \in (0,1)$ *is equivalent to a proportional TCE limit* $TCE^{\alpha,\pi}_t \leq \hat{\beta}W$*, where*

$$\beta \leq \hat{\beta} = \hat{\rho}_\alpha\left(\frac{|\kappa|\sqrt{\tau} + N^{-1}(\alpha) + \sqrt{(|\kappa|\sqrt{\tau} + N^{-1}(\alpha))^2 - 2\left(\log(1-\beta) - r\tau\right)}}{|\kappa|\sqrt{\tau}}(\sigma\sigma^\top)^{-1}\mu\right) < 1.$$

Proof. See the Appendix. □

6. Concluding Remarks

A frequently-mentioned limitation of VaR as a risk-control tool is that VaR focuses on the *probability* of large losses, but not on the *expected value* of these losses. This might induce traders subject to VaR limits to post large losses in the exceptional cases where losses exceed the VaR limit and has led several authors to propose alternatives to VaR based on the expected value of large losses. In this paper we show that this intuition, largely developed from static models, does not apply to dynamic models where the VaR is reevaluated periodically, making full use of conditioning information. Instead, in all the cases we consider, we always find that the expected value of losses and the proportional investment in risky assets are lower under a VaR constraint than they would have been without the constraint. In addition, we show that, in spite of the fact that VaR in not a coherent risk measure, risk limits formulated in terms of VaR are equivalent to to risk limits formulated in terms of TCE, which is known to be a coherent risk measure. These findings provide some theoretical support for the growing use of VaR as a risk-control tool. It remains to be seen if and to what extent they apply to models with varying price coefficients.

Appendix

Proof of Proposition 1: We have

$$\begin{aligned}
&P\left(\mathcal{W}_{t+\tau}(W_t^\pi,\pi_t) - W_t^\pi \le L \mid \mathcal{F}_t\right)\\
&= P\left(\exp\left(\left(r+\pi_t^\top\mu - \frac{1}{2}|\pi_t^\top\sigma|^2\right)\tau + \pi_t^\top\sigma(w_{t+\tau}-w_t)\right) \le 1 + \frac{L}{W_t^\pi} \mid \mathcal{F}_t\right)\\
&= P\left(\pi_t^\top\sigma(w_{t+\tau}-w_t) \le \log\left(1+\frac{L}{W_t^\pi}\right)^+ - \left(r+\pi_t^\top\mu - \frac{1}{2}|\pi_t^\top\sigma|^2\right)\tau \mid \mathcal{F}_t\right)\\
&= N\left(\frac{\log\left(1+\frac{L}{W_t^\pi}\right)^+ - \left(r+\pi_t^\top\mu-\frac{1}{2}|\pi_t^\top\sigma|^2\right)\tau}{|\pi_t^\top\sigma|\sqrt{\tau}}\right),
\end{aligned}$$

where the last equality follows from the fact that the random variable $\pi_t^\top\sigma(w_{t+\tau}-w_t)$ is conditionally normally distributed with zero mean and variance $|\pi_t^\top\sigma|^2\tau$. Thus,

$$\begin{aligned}
&P\left(\mathcal{W}_{t+\tau}(W_t^\pi,\pi_t) - W_t^\pi \le L \mid \mathcal{F}_t\right) \le \alpha\\
\iff\quad & N\left(\frac{\log\left(1+\frac{L}{W_t^\pi}\right)^+ - \left(r+\pi_t^\top\mu-\frac{1}{2}|\pi_t^\top\sigma|^2\right)\tau}{|\pi_t^\top\sigma|\sqrt{\tau}}\right) \le \alpha\\
\iff\quad & L \le W_t^\pi\left[\exp\left(\left(r+\pi_t^\top\mu-\frac{1}{2}|\pi_t^\top\sigma|^2\right)\tau + N^{-1}(\alpha)|\pi_t^\top\sigma|\sqrt{\tau}\right) - 1\right],
\end{aligned}$$

which implies

$$Q_t^{\alpha,\pi} = W_t^\pi\left[\exp\left(\left(r+\pi_t^\top\mu-\frac{1}{2}|\pi_t^\top\sigma|^2\right)\tau + N^{-1}(\alpha)|\pi_t^\top\sigma|\sqrt{\tau}\right) - 1\right].$$

Therefore,

$$VaR_t^{\alpha,\pi} = (Q_t^{\alpha,\pi})^- = W_t^\pi\left[1-\exp\left(\left(r+\pi_t^\top\mu-\frac{1}{2}|\pi_t^\top\sigma|^2\right)\tau + N^{-1}(\alpha)|\pi_t^\top\sigma|\sqrt{\tau}\right)\right]^+.$$

Similarly,

$$\begin{aligned}
&\mathrm{E}\left[(W_t^\pi - \mathcal{W}_{t+\tau}(W_t^\pi,\pi_t))1_{\{W_t^\pi-\mathcal{W}_{t+\tau}(W_t^\pi,\pi_t)\ge -Q_t^{\alpha,\pi}\}} \mid \mathcal{F}_t\right]\\
&= W_t^\pi\,\mathrm{E}\Bigg[\left(1-\exp\left(\left(r+\pi_t^\top\mu-\frac{1}{2}|\pi_t^\top\sigma|^2\right)\tau + \pi_t^\top\sigma(w_{t+\tau}-w_t)\right)\right)\\
&\qquad\qquad \times 1_{\left\{\frac{\pi_t^\top\sigma(w_{t+\tau}-w_t)}{|\pi_t^\top\sigma|\sqrt{\tau}}\le N^{-1}(\alpha)\right\}} \mid \mathcal{F}_t\Bigg]\\
&= W_t^\pi\left[\alpha - \exp\left((r+\pi_t^\top\mu)\tau\right)\int_{-\infty}^{N^{-1}(\alpha)}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(x-|\pi_t^\top\sigma|\sqrt{\tau})^2}{2}\right)dx\right]\\
&= W_t^\pi\left[\alpha - \exp\left((r+\pi_t^\top\mu)\tau\right)N\left(N^{-1}(\alpha)-|\pi_t^\top\sigma|\sqrt{\tau}\right)\right].
\end{aligned}$$

Dividing by α gives the expression for $TCE_t^{\alpha,\pi}$ in (7). $\square$

Proof of Theorem 1: The Hamilton-Jacobi-Bellman (HJB) equation for the problem in (13) is

$$\begin{aligned} 0 &= \max_{\pi}\Bigg[\frac{1}{2}V_{WW}|W\pi^\top\sigma|^2 + V_W W(r+\pi^\top\mu) + V_t \\ &\quad - \psi\Bigg(\log\left(1-\frac{\overline{VaR}}{W}\right)^+ - \left(r+\pi^\top\mu-\frac{1}{2}|\pi^\top\sigma|^2\right)\tau - N^{-1}(\alpha)|\pi^\top\sigma|\sqrt{\tau}\Bigg)\Bigg], \end{aligned} \tag{30}$$

where ψ is a Lagrangian multiplier. The first-order conditions for a maximum in (30) are:

$$V_{WW}W^2\sigma\sigma^\top\pi^* + V_W W\mu + \psi\left((\mu-\sigma\sigma^\top\pi^*)\tau + N^{-1}(\alpha)\frac{\sigma\sigma^\top\pi^*}{|\pi^{*\top}\sigma|}\sqrt{\tau}\right) = 0, \tag{31}$$

$$\psi\Bigg(\log\left(1-\frac{\overline{VaR}}{W}\right)^+ - \left(r+\pi^{*\top}\mu-\frac{1}{2}|\pi^{*\top}\sigma|^2\right)\tau - N^{-1}(\alpha)|\pi^{*\top}\sigma|\sqrt{\tau}\Bigg) = 0, \tag{32}$$

$$\log\left(1-\frac{\overline{VaR}}{W}\right)^+ - \left(r+\pi^{*\top}\mu-\frac{1}{2}|\pi^{*\top}\sigma|^2\right)\tau - N^{-1}(\alpha)|\pi^{*\top}\sigma|\sqrt{\tau} \le 0. \tag{33}$$

Rearranging equation (31) gives

$$\left[V_{WW}W^2 - \psi\left(\tau - N^{-1}(\alpha)\frac{\sqrt{\tau}}{|\pi^{*\top}\sigma|}\right)\right]\pi^* = -\left[V_W W + \psi\tau\right](\sigma\sigma^\top)^{-1}\mu.$$

Since the terms in square brackets are scalar functions of (W,t), this implies that (18) must hold for some scalar function φ. Replacing (18) in (33) gives

$$\log\left(1-\frac{\overline{VaR}}{W}\right)^+ - \left(r+\left(\varphi-\frac{1}{2}\varphi^2\right)|\kappa|^2\right)\tau - N^{-1}(\alpha)|\varphi\kappa|\sqrt{\tau} \le 0,$$

which is equivalent to

$$\varphi_\alpha^- \le \varphi \le \varphi_\alpha^+,$$

where φ_α^+ is the function defined in (14) and

$$\varphi_\alpha^-(W,t) = \frac{|\kappa|\sqrt{\tau} - N^{-1}(\alpha) - \sqrt{(|\kappa|\sqrt{\tau}-N^{-1}(\alpha))^2 - 2\left(\log\left(1-\frac{\overline{VaR}(W,t)}{W}\right)^+ - r\tau\right)}}{|\kappa|\sqrt{\tau}}.$$

Equation (31) and the complementary slackness condition (32) imply

$$(V_{WW}W^2\varphi + V_W W)\mu = 0,$$

or

$$\varphi = -\frac{V_W}{WV_{WW}}$$

when $\varphi_\alpha^- < \varphi < \varphi_\alpha^+$, and $\varphi = \varphi_\alpha^\pm$ otherwise.

Since (18) implies that optimal portfolios are combinations of the riskless asset and the growth-optimal portfolio of risky assets, $(\sigma\sigma^\top)^{-1}\mu$, the general investment problem (10) can be written equivalently as an investment problem with a single risky asset (the growth-optimal portfolio): because the constraint set for this equivalent problem is convex (see Remark 2), a standard argument implies that the value function V is (increasing and) concave. Hence, the constraint $-V_W/(WV_{WW} > \varphi_\alpha^-$ is never binding (because φ_α^- is nonpositive). This establishes the equality in (17).

Finally, (15) follows from substituting (18) and (17) in (30). □

Proof of Lemma 1: It is easily verified that $\hat{\rho}_{\hat{\alpha}}(0) = 1 - e^{r\tau} < 0$ and that

$$\frac{\partial}{\partial\varphi}\hat{\rho}_{\hat{\alpha}}(\varphi(\sigma\sigma^\top)^{-1}\mu) = \exp\left(\left(r + \varphi|\kappa|^2\right)\tau\right)\frac{N\left(N^{-1}(\hat{\alpha}) - |\varphi||\kappa|\sqrt{\tau}\right)|\kappa|\sqrt{\tau}}{\hat{\alpha}}f(\varphi)$$

for $\varphi \neq 0$, where

$$f(\varphi) = \frac{N'\left(N^{-1}(\hat{\alpha}) - |\varphi||\kappa|\sqrt{\tau}\right)}{N\left(N^{-1}(\hat{\alpha}) - |\varphi||\kappa|\sqrt{\tau}\right)}\operatorname{sign}(\varphi) - |\kappa|\sqrt{\tau}.$$

Clearly, $f(\varphi) < 0$ on $(-\infty, 0)$. Moreover, $f(\varphi)$ is monotonically increasing on $(0, +\infty)$, with $\lim_{\varphi\to+\infty} f(\varphi) = +\infty$. Letting

$$\varphi^* = \inf\{\varphi \geq 0 : f(\varphi) \geq 0\},$$

this implies that $\hat{\rho}_{\hat{\alpha}}(\varphi(\sigma\sigma^\top)^{-1}\mu)$ is a monotonically decreasing function of φ on $(-\infty, 0)$ and a monotonically increasing function on $(\varphi^*, +\infty)$, with $\hat{\rho}_{\hat{\alpha}}(\varphi(\sigma\sigma^\top)^{-1}\mu) < 0$ on $[0, \varphi^*]$.

Since

$$\lim_{\varphi\to-\infty}\hat{\rho}_{\hat{\alpha}}(\varphi(\sigma\sigma^\top)^{-1}\mu) = 1 - \lim_{\varphi\to-\infty}\exp\left(\left(r + \varphi|\kappa|^2\right)\tau\right)\frac{N\left(N^{-1}(\hat{\alpha}) + \varphi|\kappa|\sqrt{\tau}\right)}{\hat{\alpha}} = 1$$

and

$$\lim_{\varphi\to+\infty}\hat{\rho}_{\hat{\alpha}}(\varphi(\sigma\sigma^\top)^{-1}\mu) = 1 - \lim_{\varphi\to+\infty}\exp\left(\left(r + \varphi|\kappa|^2\right)\tau\right)\frac{N\left(N^{-1}(\hat{\alpha}) - \varphi|\kappa|\sqrt{\tau}\right)}{\hat{\alpha}} = 1,$$

the claim immediately follows. □

Proof of Proposition 2: Letting

$$\varphi^* = \frac{|\kappa|\sqrt{\tau} + N^{-1}(\alpha) + \sqrt{(|\kappa|\sqrt{\tau} + N^{-1}(\alpha))^2 + 2r\tau}}{|\kappa|\sqrt{\tau}}$$

denote the positive root of the equation $\rho_\alpha(\varphi(\sigma\sigma^\top)^{-1}\mu) = 0$, it is easily verified that

$$\frac{\partial}{\partial\varphi}\rho_\alpha(\varphi(\sigma\sigma^\top)^{-1}\mu) = |\kappa|\sqrt{\tau}\Big(1 - \rho_\alpha(\varphi(\sigma\sigma^\top)^{-1}\mu)\Big)\Big(|\kappa|\sqrt{\tau}\varphi - (|\kappa|\sqrt{\tau} + N^{-1}(\alpha))\Big) > 0$$

for $\varphi > \varphi^*$. Therefore, if α and $\overline{VaR}$ satisfy (26) and (27), we have

$$\overline{VaR}(W,t) = W\rho_\alpha(\hat{\varphi}^+_{\hat{\alpha}}(W,t)(\sigma\sigma^\top)^{-1}\mu) \geq W\rho_\alpha(\varphi^*(\sigma\sigma^\top)^{-1}\mu) = 0$$

and

$$\varphi^+_\alpha(W,t) = \sup\Big\{\varphi \geq 0 : W\rho_\alpha(\varphi(\sigma\sigma^\top)^{-1}\mu) \leq \overline{VaR}(W,t)\Big\} = \hat{\varphi}^+_{\hat{\alpha}}(W,t).$$

Similarly, if $\hat{\alpha}$ and $\overline{TCE}$ satisfy (28) and (29), we have

$$\overline{TCE}(W,t) = W\hat{\rho}_{\hat{\alpha}}(\varphi^+_\alpha(W,t)(\sigma\sigma^\top)^{-1}\mu) \geq W\hat{\rho}_{\hat{\alpha}}(\Phi^+_{\hat{\alpha}}(0)(\sigma\sigma^\top)^{-1}\mu) = 0$$

and

$$\hat{\varphi}^+_{\hat{\alpha}}(W,t) = \sup\Big\{\varphi \geq 0 : W\hat{\rho}_{\hat{\alpha}}(\varphi(\sigma\sigma^\top)^{-1}\mu) \leq \overline{TCE}(W,t)\Big\} = \varphi^+_\alpha(W,t). \qquad \square$$

Proof of Corollary 2: When $\overline{VaR}(W,t) = \beta W$, (14) gives $\varphi^+_\alpha(W,t) = \varphi^*$ for all $(W,t) \in \mathbb{R}^+ \times [0,T]$, where

$$\varphi^* = \frac{|\kappa|\sqrt{\tau} + N^{-1}(\alpha) + \sqrt{(|\kappa|\sqrt{\tau} + N^{-1}(\alpha))^2 - 2\left(\log(1-\beta) - r\tau\right)}}{|\kappa|\sqrt{\tau}}$$

Since (8) implies $\rho_\alpha(\varphi(\sigma\sigma^\top)^{-1}\mu) \leq \hat{\rho}_\alpha(\varphi(\sigma\sigma^\top)^{-1}\mu)$ for all $\varphi \in \mathbb{R}$, we have

$$\begin{aligned}
\inf_{(W,t)\in\mathbb{R}^+\times[0,T]} \varphi^+_\alpha(W,t) &= \varphi^* \\
&= \sup\Big\{\varphi \in \mathbb{R} : \rho_\alpha(\varphi(\sigma\sigma^\top)^{-1}\mu) \leq \beta\Big\} \\
&\geq \sup\Big\{\varphi \in \mathbb{R} : \hat{\rho}_\alpha(\varphi(\sigma\sigma^\top)^{-1}\mu) \leq \beta\Big\} \\
&\geq \sup\Big\{\varphi \in \mathbb{R} : \hat{\rho}_\alpha(\varphi(\sigma\sigma^\top)^{-1}\mu) \leq 0\Big\} \\
&= \Phi^+_\alpha(0).
\end{aligned}$$

Therefore, the condition (28) is satisfied with $\hat{\alpha} = \alpha$. It then immediately follows from Proposition 2 that the proportional VaR limit $VaR^{\alpha,\pi}_t \leq \beta W^\pi_t$ is equivalent to the proportional TCE limit $TCE^{\alpha,\pi}_t \leq \hat{\beta} W^\pi_t$, where $\hat{\beta} = \hat{\rho}_\alpha(\varphi^*(\sigma\sigma^\top)^{-1}\mu)$. Moreover, it follows from (8) that

$$\beta = \rho_\alpha(\varphi^*(\sigma\sigma^\top)^{-1}\mu) \leq \hat{\rho}_{\hat{\alpha}}(\varphi^*(\sigma\sigma^\top)^{-1}\mu) = \hat{\beta}$$

and

$$\hat{\beta} = \hat{\rho}_{\hat{\alpha}}(\varphi^*(\sigma\sigma^\top)^{-1}\mu) < 1. \qquad \square$$

References

AHN, D.-H., J. BOUDOUKH, M. RICHARDSON AND R.F. WHITELAW, 1999, "Optimal Risk Management Using Options", *Journal of Finance* **54**, 359–375.

ALEXANDER, G. AND A. BAPTISTA, 2000, "Economic Implications of Using a Mean-VaR Model for Portfolio Selection: A Comparison with Mean-Variance Analysis", working paper, University of Minnesota.

ARTZNER, P., F. DELBAEN, J.-M. EBER AND D. HEATH, 1999, "Coherent Measures of Risk", *Mathematical Finance* **9**, 203–228.

BASAK, S. AND A. SHAPIRO, 2001, "Value-at-Risk Based Risk Management: Optimal Policies and Asset Prices", *Review of Financial Studies* **14**, 371–405.

CVITANIĆ, J. AND I. KARATZAS, 1992, "Convex Duality in Constrained Portfolio Optimization", *Annals of Applied Probability* **2**, 767–818.

DUFFIE, D AND J. PAN, 1997, "An Overview of Value at Risk", *Journal of Derivatives* **4**, 7–49.

EMBRECHTS, P., 1999, "Extreme Value Theory as a Risk Management Tool", *North American Actuarial Journal*.

EMMER, S., C. KLÜPPELBERG AND R. KORN, 2001, "Optimal Portfolios with Bounded Capital-at-Risk", *Mathematical Finance*, forthcoming.

JORION, P., 2001, *Value at Risk*, McGraw-Hill, New York.

KARATZAS, I. AND S.E. SHREVE, 1988, *Brownian Motion and Stochastic Calculus*, Springer-Verlag, New York.

KAST, R., E. LUCIANO AND L. PECCATI, 1999, "Value-at-Risk as a Decision Criterion", working paper, University of Turin.

KUSHNER, H.J. AND P.G. DUPUIS, 1992, *Numerical Methods for Stochastic Control Problems in Continuous Time*, Springer-Verlag, New York.

PFLUG, G., 2000, "Some Remarks on the Value-at-Risk and the Conditional Value-at-Risk", in: S. Uryasev (ed.), *Probabilistic Constrained Optimization: Methodology and Applications*, Kluwer, Boston.

ROCKAFELLAR, R.T. AND S. URYASEV, 2001, "Optimization of Conditional Value-at-Risk", *Journal of Risk*, forthcoming.

VORST, T., 2001, "Optimal Portfolios under a Value at Risk Constraint", in: C. Casacuberta, R.M. Miró-Roig, J. Verdera and S. Xambó (eds.), *Proceedings of the European Congress of Mathematics, Barcelona, July 10-14, 2000*, Birkhäuser, Basel, forthcoming.